Hartle Solutions week 9

8.2

Solution:

a)

$$g_{AB} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix}$$

$$g^{AB} = \begin{pmatrix} R^{-2} & 0 \\ 0 & (R\sin\theta)^{-2} \end{pmatrix}$$

$$\Gamma^1_{22} = -\sin\theta\cos\theta \quad , \quad \Gamma^2_{12} = \cot\theta$$

all the rest are zero.

b) Orient coordinates so that the great circle lies along the equator $\theta = \pi/2$. The equation of the great circle is then $\theta = \pi/2, \phi = S/a$ where S is the distance around and $dx^A/dS = (0, 1/a)$. The geodesic equation is then

$$\frac{d^2x^A}{dS^2} = -\Gamma^A_{22}\frac{1}{a^2}$$

Evidently the left hand side vanishes, and the right hand side vanishes because the relevant Christoffel symbols vanish at $\theta = \pi/2$.

8.3

Solution:

a) The Lagrangian $L(\dot{x}^\alpha, x^\alpha) \equiv [-g_{\alpha\beta}(x)\dot{x}^\alpha\dot{x}^\beta]^{1/2}$ defined by the principle of extremal proper time [*cf.* (8.10)] is for this problem

$$L\left(\dot{t}, \dot{r}, \dot{\phi}, r\right) = \left[\left(1 - \frac{2M}{r}\right)(\dot{t})^2 - \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2\right]^{\frac{1}{2}}$$

where $\dot{x}^\alpha = dx^\alpha/d\sigma$.

b) The components of the geodesic equations are [*cf.* (8.9)]

$$-\frac{d}{d\tau}\left[\left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}\right] = 0$$

$$-\frac{d}{d\tau}\left[-r^2\frac{d\phi}{d\tau}\right] = 0$$

$$-\frac{d}{d\tau}\left[\left(1 - \frac{2M}{r}\right)^{-1}\frac{dr}{d\tau}\right] + \frac{M}{r^2}\left(\frac{dt}{d\tau}\right)^2$$

$$+\left(1 - \frac{2M}{r}\right)^{-2}\frac{M}{r^2}\left(\frac{dr}{d\tau}\right)^2 - r\left(\frac{d\phi}{d\tau}\right)^2 = 0$$

c) From these we find the following non-vanishing Christoffel symbols

$$\Gamma^t_{tr} = \left(1 - \frac{2M}{r}\right)^{-1} \frac{M}{r^2}\,, \quad \Gamma^r_{\phi\phi} = -(r-2M)\,,$$

$$\Gamma^r_{tt} = \left(1 - \frac{2M}{r}\right) \frac{M}{r^2}\,, \quad \Gamma^\phi_{\phi r} = \frac{1}{r}\,,$$

$$\Gamma^r_{rr} = -\left(1 - \frac{2M}{r}\right)^{-1} \frac{M}{r^2}\,.$$

8-9. *Consider the two-dimensional spacetime with the line element*

$$ds^2 = -X^2 dT^2 + dX^2$$

Find the shapes $X(T)$ of all the timelike geodesics in this spacetime.

Solution: There are several ways to do this problem. The most direct is to note that the metric is independent of T and therefore has a Killing vector $\xi^A = (1,0)$. The quantity

$$-\boldsymbol{\xi}\cdot\mathbf{u} = X^2\left(\frac{dT}{d\tau}\right) \equiv e \qquad (1)$$

is conserved and is a first integral of the geodesic equations. Another first integral is supplied by $\mathbf{u}\cdot\mathbf{u} = -1$ which is

$$-X^2\left(\frac{dT}{d\tau}\right)^2 + \left(\frac{dX}{d\tau}\right)^2 = -1$$

Substituting for $dT/d\tau$ from (1) gives

$$\frac{dX}{d\tau} = \pm\left(\frac{e^2}{X^2} - 1\right)^{\frac{1}{2}}$$

To compute the shape $t(x)$ note that

$$\frac{dT}{dX} = \frac{dT/d\tau}{dX/d\tau} = \pm\frac{e}{X^2}\left(\frac{e^2}{X^2} - 1\right)^{-\frac{1}{2}}$$

This can be integrated to find

$$T(X) = \pm\ \cosh^{-1}\left(\frac{e}{X}\right) + T_0\,.$$

This is a two parameter family of all timelike geodesics.

Another way of doing this problem is to write out the geodesic equations and deduce the integral (1). Yet another is to note that $x = X\cosh T$, $t = X\sinh T$ transforms flat spacetime into this form, and transforms back the straight lines in flat spacetime.

8.12

Solution:

a) The distance along an x-constant line to a point on the x-axis is $\int_0 dy/y$ which diverges. Similarly, the distance along any other curve to a point on the x-axis will diverge.

b) The non-vanishing Christoffel symbols are:

$$-\Gamma^x_{xy} = \Gamma^y_{xx} = -\Gamma^y_{yy} = 1/y \ .$$

The geodesic equations are:

$$\frac{d^2x}{dS^2} = \frac{2}{y}\frac{dx}{dS}\frac{dy}{dS}$$

$$\frac{d^2y}{dS^2} = -\frac{1}{y}\left(\frac{dx}{dS}\right)^2 + \frac{1}{y}\left(\frac{dy}{dS}\right)^2 \ .$$

c) The first geodesic equation in (b) can be written

$$y^2\frac{d}{dS}\left(\frac{1}{y^2}\frac{dx}{dS}\right) = 0 \ .$$

This can be integrated to give

$$\frac{dx}{dS} = \frac{y^2}{r} \tag{1}$$

for any constant r. This is one integral of the geodesic equations. Another is supplied by the normalization condition

$$\frac{1}{y^2}\left[\left(\frac{dx}{dS}\right)^2 + \left(\frac{dy}{dS}\right)^2\right] = 1 \ .$$

Using (1) we have,

$$\frac{dy}{dS} = \pm\left[y^2 - \frac{y^4}{r^2}\right]^{\frac{1}{2}} \tag{2}$$

To find the shape of the geodesics we compute (for $dy/dS > 0$)

$$\frac{dx}{dy} = \frac{dx/dS}{dy/dS} = \frac{y}{\sqrt{r^2 - y^2}} \ .$$

The integral of this is

$$(x - x_0)^2 + y^2 = r^2$$

where x_0 is a constant. This is a circle of radius r centered at $(x_0, 0)$ on the x-axis. Vertical lines correspond to the limit in which r becomes infinite.

d) Eq (2) can be solved to yield

$$y(S) = \frac{r}{\cosh(S)}$$

and then using this in (1) gives

$$x(S) = r \tanh(S) \; .$$

You can check that $x^2 + y^2 = r^2$.

9.1

Solution: The area of a sphere of Schwarzschild radius r is $4\pi r^2$. The Schwarzschild radii of the inside and outside of the shell are therefore $6M$ and $10M$. The radial distance between these shells is thus,

$$\begin{aligned} d = \int ds &= \int_{6M}^{10M} dr \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} \\ &= \left\{ r\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} + M \log \left[r - M + r\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \right] \right\} \Bigg|_{6M}^{10M} \\ &= 4.64M \end{aligned}$$

9.2

Solution: Electrons and positrons annihilate to produce two gamma rays. (Producing just one is forbidden by conservation of linear momentum.) Therefore, in the center of mass frame of the electron and positron each gamma ray will have an energy of $m_e c^2 \approx .5MeV$, the rest energy of an electron. The gamma ray is redshifted when it reaches infinity by an amount

$$\begin{aligned} E_\infty &= m_e c^2 \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} \\ &\approx .5MeV \left[1 - \frac{2(2.5M_\odot)(1.474 \text{ km}/M_\odot)}{10 \text{ km}}\right]^{\frac{1}{2}} \\ &\approx .26 \text{ MeV} \end{aligned}$$

9.3

Solution:

a) From the principle of equivalence, the relationship between E and P must be exactly the same as it is for a flat space:

$$E^2 - P^2 = m^2$$

where m is the proton mass.

b) Measured quantities correspond to projections on the orthonormal basis $\mathbf{e}_{\hat{\alpha}}$ associated with the observers' laboratory. If one of the spacelike vectors $\mathbf{e}_{\hat{r}}$ is oriented in the r direction

$$E = -\mathbf{p}\cdot\mathbf{e}_{\hat{0}} \quad , \quad P = \mathbf{p}\cdot\mathbf{e}_{\hat{r}}$$

The Schwarzschild coordinate components of $\mathbf{e}_{\hat{0}}$ and $\mathbf{e}_{\hat{r}}$ are easily worked out from $\mathbf{e}_{\hat{\alpha}}\cdot\mathbf{e}_{\hat{\beta}} = \eta_{\alpha\beta}$ and the requirement that $\mathbf{e}_{\hat{0}}$ point in the t direction (stationary) and $\mathbf{e}_{\hat{r}}$ in the r direction

$$\mathbf{e}_{\hat{0}} = \left(\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}}, 0, 0, 0\right) ,$$

$$\mathbf{e}_{\hat{r}} = \left(0, \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}}, 0, 0\right)$$

Thus

$$E = -\mathbf{p}\cdot\mathbf{e}_{\hat{0}} = \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} p^t$$

and

$$P = (1 - 2M/R)^{-\frac{1}{2}}\, p^r .$$

From these two relations p^t and p^r can be calculated in terms of P and E.