This is an open book test. You may refer to your notes and textbooks.

1. Sketch examples of phase portraits before, at, and after each of the following possible bifurcations:

   (a) An infinite period bifurcation.

   (b) A degenerate Hopf bifurcation.

2. A system is given in polar form below

$$\dot{r} = r(r^2 - \mu r + 1) \qquad \text{and} \qquad \dot{\theta} = 1$$

   For what value of the parameter μ is there a bifurcation? What type of bifurcation is it? Sketch the phase portrait before and after the bifurcation.

3. The following systems represents a model for the autocatalytic chemical reaction that is thought to govern circadian rhythms in the human body

$$\begin{aligned} \dot{x} &= 1 - axy \\ \dot{y} &= axy - \frac{by}{1+y} \end{aligned}$$

x and y are non-dimensionality concentrations of two chemicals and a and b are positive rates.

(a) Find the fixed point of the system, and determine the conditions on the rate constants a and b so that the fixed point is in the first quadrant of phase space.

(b) Use the Jacobian to classify the fixed point. In particular show that there is a critical value $a = a_c$ at which there is a bifurcation in stability. Express a_c in terms of b.

(c) Sketch the nullclines assuming the condition in part (a) applies. (Take $b = 2$ if it makes your life easier.) Construct as complete a trapping region as possible. Most trapping boundaries should be straightforward to find, however the region near the origin along the x axis may cause difficulty. You are not required to find the trapping boundary here but bonus points will be awarded it you do.

(d) What type of bifurcation do your answers to (b) and (c) suggest occur at $a = a_c$.

4. Consider the following system.

$$\dot{x} = y\,, \qquad \dot{y} = x(1-z) - ay \qquad \dot{z} = -z + x^2$$

(a) Find and classify the fixed points of this system.

(b) There is a Hopf bifurcation in this system. Determine the critical value of a_c at which it occurs.

(c) Estimate the period of the oscillations near the critical point $a = a_c$.