This test is due on Monday, March 2nd at 9:00 am. You may refer to your notes and textbooks, but you must not consult with other people.

1. Consider the system: $\dfrac{\mathrm{d}x}{\mathrm{d}t} = x - x^2 + y + y^2 \qquad \dfrac{\mathrm{d}y}{\mathrm{d}t} = x - y + 2xy$

   (a) Is the system a gradient system? Explain.

   (b) Is this system a Hamiltonian system? Explain.

   (c) Find a quantity $F(x, y)$ that is conserved along solution curves in the phase plane.

   (d) Find an expression for the curve in phase space passing through the point (2,0).

2. Consider the following differential equation modeling the motion of body as it falls under the influence of gravity and air resistance:

$$\ddot{x} = -g + \epsilon\dot{x}^2 \qquad \text{and} \qquad x(0) = 5,\ \dot{x}(0) = 0$$

   were g is the acceleration due to gravity and ϵ is a dimensionless parameter. Assume $\epsilon \ll 1$.

   (a) Expand the differential equation perturbatively by writing x as $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 \cdots$ and hence find linear differential equations for each of the terms x_0, x_1 and x_2.

   (b) Solve the differential equations in part (a), and write down a perturbative solution for x.

3. Explain why the following systems don't have closed orbits.

   (a)
$$\begin{aligned} \dot{x} &= y + x^3 \\ \dot{y} &= x + y + y^3 \end{aligned}$$

   (b)
$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -(1 + x^2)y - x \end{aligned}$$

4. Consider the system: $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 2y - x^2 \qquad \dfrac{\mathrm{d}y}{\mathrm{d}t} = 2x - y$

   (a) Find all the equilibrium points for this system.

   (b) Using the Jacobian matrix determine the type of each equilibrium point.

   (c) Sketch nullclines for this system, indicate the direction of the vector field on each and then sketch a phase portrait for the system.

5. The phase portraits below correspond to systems of coupled differential equations in x and y. For each portrait indicate which, if any, of the following descriptions could apply to the system. For each system explain why each of the possible descriptions is or is not appropriate.
   (a) Linear System (b) Gradient System (c) Hamiltonian System