Simulating Gravity in a Rotating Frame

A space ship is in the shape of a huge rotating disc with radius R. It is designed to rotate with angular velocity ω, to simulate gravity directed outward at its rim. Two space travelers are on this disc. Alice is in the center of the ring, and Bob is at the rim.

1. Suppose an outside observer, in an inertial frame just above the center of the disc measures a time Δt between two events. By finding the speed of those two observers relative to this observer find Δt_A and Δt_B the times measured by Alice and Bob. Hence find a relationship between the times measured by Alice and Bob for the two events.

2. Now consider the same situation from the rotating frame of the disc. Bob and Alice are relatively at rest in this frame, so experience no relativistic time dilation. However, because Bob is accelerating (according to outside observers), he feels like he is in a gravitational field. What is the magnitude and direction of this field, as a function of distance from the center of the disc, r? Hence find the the gravitational potential as a function of r, assuming it is zero at the center. Use this potential, and gravitational time dilation to show that the relationship between Δt_A and Δt_B is the same as in the previous question, to first order in ωr.

3. If observers in the rotating disc experience gravity, they must experience curved spacetime. We can find the metric for this spacetime by considering the line element for the inertial observer in cylindrical coordinates:
$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2\,,$$
and then make the observation that if Alice measures an angle θ' in her frame, then the outside observer will measure an angle $\theta = \theta' + \omega t$. Hence show that the line element for Alice and Bob is
$$ds^2 = -(1-\omega^2 r^2)dt^2 + 2\omega r^2 dt d\phi' + dr^2 + r^2 d\phi'^2 + dz^2\,.$$
The line element has a cross term in it, so the metric is not diagonal.

4. Alice wishes to time how long it takes for light to travel through an optical fiber around the circumference of the disk. She asks Bob to send one beam clockwise and one counter clockwise and times how long it takes for the beams to return to Bob's position. Take $r = R$, and $z = 0$ in the line element and then show that the counter clockwise and clockwise moving light beams have angular velocity given by
$$\frac{\mathrm{d}\phi'}{\mathrm{d}t} = \omega \pm \frac{1}{R}\,.$$
Hence show that the time it takes light to move clockwise and the time it takes light to move counter clockwise are different and are given by
$$\Delta t_+ = \frac{2\pi R}{1+R\omega} \qquad \text{and} \qquad \Delta t_- = \frac{2\pi R}{1-R\omega}\,.$$

5. The above result is called the Sagnac effect, and provides a way for observers in rotating frames to tell that they are in a non-inertial frame. By measuring the difference in time between the two beams Alice can determine ω. Show that if $\delta t = t_- - t_+$ then
$$\omega = \frac{2\pi}{\delta t} - \sqrt{\left(\frac{2\pi}{\delta t}\right)^2 - \frac{1}{R^2}} \approx \frac{\delta t}{4\pi R^2}$$