1. Find the inverse of the matrix $B = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$. Check that $B^{-1}B = I$.

   $\text{Det}A = 2(1) - (3)(4) = -10$, so $A^{-1} = -\frac{1}{10}\begin{pmatrix} 1 & -3 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} -0.1 & 0.3 \\ 0.4 & -0.2 \end{pmatrix}$

2. Finding inverses for matrices that are larger than 2×2 is also possible, but harder. However, it is possible to check whether a matrix is an inverse by showing that $A^{-1}A = I$. Show that the inverse of the matrix $A = \begin{pmatrix} 4 & 8 & 3 \\ 3 & 5 & 1 \\ 1 & 4 & 3 \end{pmatrix}$ is $A^{-1} = \begin{pmatrix} 11 & -12 & -7 \\ -8 & 9 & 5 \\ 7 & -8 & -4 \end{pmatrix}$.

   $$A^{-1}A = \begin{pmatrix} 11 & -12 & -7 \\ -8 & 9 & 5 \\ 7 & -8 & -4 \end{pmatrix}\begin{pmatrix} 4 & 8 & 3 \\ 3 & 5 & 1 \\ 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

3. A matrix is *singular* if it does not have an inverse. This happens when its determinant is zero. Find the values of k such that the following matrix is singular $A = \begin{pmatrix} 4+k & 3 \\ -1 & k \end{pmatrix}$

   $\text{Det}A = (4+k)(k) - (3)(-1) = 4k + k^2 + 3$. To be singular we want the $\text{Det}A = 0$. So $k^2 + 4k + 3 = 0 \Rightarrow (k+3)(k+1) = 0$. So $k = -1$ or $k = -3$. Thus the following matrices are singular $\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}$.

4. In a particular patch of forest there are only maple and fir trees. Suppose 2% of fir die each year and 4% of maple die each year. Whenever a tree dies a new tree grow in its place. Suppose that 80% of new trees are maple and 20% of new trees are fir. If we let f represent the number of firs and m represent the number of maples, write down a system of difference equations for the growth of trees in the forest. Now express these equations as a single matrix equation. Compute M^{100} and use this matrix to determine the distribution of trees after 100 years if you start with 500 of each type.

   $$\begin{aligned} m_{t+1} &= 0.96m_t + 0.8(0.04m_t + 0.02f_t) = 0.992m_t + 0.016f_t \\ f_{t+1} &= 0.98f_t + 0.2(0.04m_t + 0.02f_t) = 0.008m_t + 0.984f_t \end{aligned}$$

   So the matrix equation is $\begin{pmatrix} m_{t+1} \\ f_{t+1} \end{pmatrix} = \begin{pmatrix} 0.992 & 0.016 \\ 0.008 & 0.984 \end{pmatrix}\begin{pmatrix} m_t \\ f_t \end{pmatrix}$.

5. In this example you will learn how to find the eigenvalues and eigenvectors of a matrix. Consider the matrix $M = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$. We want to find λ and $\vec{v}$ such that $M\vec{v} = \lambda\vec{v}$. This means that

$$\begin{aligned} M\vec{v} - \lambda\vec{v} &= 0 \\ M\vec{v} - \lambda I\vec{v} &= 0 \\ (M - \lambda I)\vec{v} &= 0 \end{aligned}$$

where we introduce the identity matrix I so that we can factor out the vector $\vec{v}$. Since we assume that $\vec{v}$ is not zero, this means that the matrix $M - \lambda I$ must be singular. That is $\mathrm{Det}(M - \lambda I) = 0$.

(a) Show that $M - \lambda I = \begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}$

$$M - \lambda I = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}.$$

(b) Find an expression for $\mathrm{Det}(M - \lambda I)$ (It should be a quadratic polynomial).
$\mathrm{Det}(M - \lambda I) = (4-\lambda)(3-\lambda) - (2)(1) = 12 - 7\lambda + \lambda^2 - 2 = \lambda^2 - 7\lambda + 10.$

(c) Using the result above, solve the equation $\mathrm{Det}(M - \lambda I) = 0$, for λ.
$\mathrm{Det}(M - \lambda I) = 0 \Rightarrow \lambda^2 - 7\lambda + 10 = 0 \Rightarrow (\lambda - 5)(\lambda - 2) = 0 \Rightarrow \lambda = 5 \quad \text{or} \quad \lambda = 2.$

(d) You should have two values for λ. For each eigenvalue find the corresponding eigenvector $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ by finding suitable x and y so that $\begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$. Note: If $\vec{v}$ is an eigenvector then any multiple of $\vec{v}$ is also an eigenvector, so you have some freedom to choose either x or y to make your vectors simple.
First we take $\lambda = 5$.

$$\begin{pmatrix} 4-5 & 2 \\ 1 & 3-5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow -x + 2y = 0 \quad \text{or} \quad x - 2y = 0 \Rightarrow x = 2y\,.$$

So $\vec{v} = \begin{pmatrix} 2y \\ y \end{pmatrix} = y \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for any non-zero y (easiest to take $y = 1$). Now we take $\lambda = 2$.

$$\begin{pmatrix} 4-2 & 2 \\ 1 & 3-2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow 2x + 2y = 0 \quad \text{or} \quad x + y = 0 \Rightarrow x = -y\,.$$

So $\vec{v} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ for any non-zero y.

6. The following is the Leslie matrix for a certain animal population divided into two five year classes

$$M = \begin{pmatrix} 2/3 & 3/2 \\ 2/9 & 0 \end{pmatrix}$$

(a) Find the eigenvalues λ_1 and λ_2 for this matrix given that the eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} 9 \\ 2 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$M\vec{v} = \lambda\vec{v}$ if $\vec{v}$ is an eigenvector, by definition. But

$$M\vec{v}_1 = \begin{pmatrix} 2/3 & 3/2 \\ 2/9 & 0 \end{pmatrix}\begin{pmatrix} 9 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \end{pmatrix}.$$

Therefore $\lambda_1 = 1$. Similarly

$$M\vec{v}_2 = \begin{pmatrix} 2/3 & 3/2 \\ 2/9 & 0 \end{pmatrix}\begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2/3 \end{pmatrix} = (-1/3)\begin{pmatrix} -3 \\ 2 \end{pmatrix},$$

so $\lambda_2 = -1/3$.

(b) If the initial population of the two classes is

$$\vec{P}_o = \begin{pmatrix} 30 \\ 60 \end{pmatrix}$$

find the scalars c_1 and c_2 such that

$$\vec{P}_o = c_1\vec{v}_1 + c_2\vec{v}_2$$

We get the two equations

$$\begin{array}{lrcl} (1) & 30 & = & 9c_1 - 3c_2 \\ (2) & 60 & = & 2c_1 + 2c_2 \end{array}$$

If we take equation $2\times$ eqn(1) plus $3\times$ eqn(2) we can eliminate c_2. This gives us

$$60 + 180 = 18c_1 + 6c_1 - 6c_2 + 6c_2 \;\Rightarrow\; 240 = 24c_1 \;\Rightarrow c_1 = 10\ .$$

Substituting into equation (2) gives $60 = 2(10) + 2c_2 \;\Rightarrow\; c_2 = 40/2 = 20$.

(c) Write down the general solution for $\vec{P}_t$ in terms of eigenvectors and eigenvalues.

$$\vec{P}_t = 10(1)^t\begin{pmatrix} 9 \\ 2 \end{pmatrix} + 20(-1/3)^t\begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

.

(d) Describe the long term behavior of the population. That is, describe the dominant solution, and the stable stage distribution. Sketch a graph of each population group as a function of time, clearly showing the transient behavior on your graph.
The dominant eigenvalue is $\lambda_1 = 1$ and $(1)^t = 1$, so the dominant solution is an equilibrium equal to $10\begin{pmatrix} 9 \\ 2 \end{pmatrix} = \begin{pmatrix} 90 \\ 20 \end{pmatrix}$. The stable stage distribution is 9:2. The other eigenvalue is $(-1/3)$ so the transient behavior is an oscillating decay. $x_t = 90 - 60(-1/3)^t$ and $y_t = 20 + 40(-1/3)^t$. So x_t starts at $x_0 = 30$ and then oscillates above and below $x = 90$, gradually approaching this equilibrium value. Similarly $y_t =$ starts at $y_0 = 60$ and oscillates below and then above $y = 20$, gradually approaching this equilibrium value.

7. The first row of a Leslie matrix shows the birthrates for individuals in each class. The main sub-diagonal of the matrix gives the survival rates of each class (or the rate at which members transition from one class to the next.) The following data ia from page 55 or your text book, for a Leslie model of the US population, for *females* between 0 and 50 years old, divided into 10 equally sized classes. The first row is:

$$(0.000, 0.010, 0.878, 0.3487, 0.4761, 0.3377, 0.1833, 0.0761, 0.0174, 0.0010)$$

The subdiagnoal is:

$$(0.9966, 0.9983, 0.9979, 0.9968, 0.9961, 0.9947, 0.9923, 0.9987, 0.9831)$$

(a) What does the second number in the first row mean? Explain why it is not zero, but the first number is.
This is the birth rate for 5-10 year-olds in the US for a five year period. This is non zero because in that five year period these children age, and some of them pass puberty and have children. The first number is zero because 0-5 year-olds do not age enough in 5 years to have any children.

(b) Why would the first entry in the sub-diagonal be smaller than the second entry? What does this entry mean?
The first entry is the fraction of 0-5 year-olds who reach the second class in five years. Because of infant mortality this survival rate is less than the survival rate for 5-10 year-olds.

(c) Notice that the seventh entry in the sub-diagonal is less than the neighboring values. Can you think of a reason why?
This is is the survival rate for the 30-35 age group. This is the peak child-rearing age and due to the risk of death during child birth this number is slightly lower.

(d) Why is it reasonable to only included females up to 50 in this model?
Very few woman have children after 50, so including those individuals will not have an impact on the growth rate predicted by the model.