We often consider outcomes of events that are combinations of two simpler events. For example, getting a double six when throwing two dice is a combination of getting a six on the first dice and a six on the second. The probability of combined events follow special rules when certain conditions apply. Here are some special cases.

Independent Events: Two events are independent if the occurrence of one does no affect the probability of the other occurring. (e.g. Getting a heads on one coin will not affect the probability of getting heads on the second.)

Mutually Exclusive Events: Two events are mutually exclusive if the occurrence of one means the other cannot occur. (e.g. When tossing a die, the events A="getting a six" and B="getting a 1" are mutually exclusive.)

Exhaustive Events: Two events are exhaustive if whenever one of the events doesn't occur the other one must. (e.g. When we toss a coin heads and tails are exhaustive events).

1. Decide whether or not the following events are independent, mutually exclusive, and/or exhaustive.

   (a) The weather is nice; I walk to work.
   If the weather is nice I am more likely to walk to work, so they are not independent. It is possible that I will walk to work when it is nice so they are not mutually exclusive. If it is not nice it is possible that I will not work so they are not exhaustive.

   (b) I cut an Ace; you cut a King.
   These events are independent (one does not affect the other), not mutually exclusive (they can both happen) and not mutually exclusive (there are other cards to draw).

   (c) Mrs Smith has cold; Mr Smith has a cold.
   These events are not independent – if the wife has a cold the husband is more likely to. They are not mutually exclusive – they can both have a cold at the same time. They are not mutually exclusive – its possible for neither to have a cold.

   (d) Mrs Smith has a tooth ache, Mr Smith has tooth ache.
   Similar to above except they are independent events – Mrs Smith having a tooth ache won't make Mr Smith more likely to have one.

   (e) I get a number less than 4 on a die toss; I get a number greater than 2 on the same die toss.
   Knowing that the number on a dice toss is less than 4 changes the probability that the number is greater than 2 (it becomes less likely) so the events are not independent. They are not mutually exclusive since I could get a 3 which is both less than 4 and greater than 2. They are exhaustive – all scores on a die are either less than 4 or greater than 2.

   (f) The base at a DNA site is adenine, The base at the same site is a pyrimidine.
   They are not independent. If I get an adenine I know with certainty that it is not a pyrimidine. They are mutually exclusive – an adenine is not a pyrimidine. They are not exhaustive – guanine is not a pyrimidine and also is not adenine.

2. Can two events be mutually exclusive and independent? If so, given an example. If not explain why not.
No. If two events are mutually exclusive then knowing that one occurs means you know with certainty that the other hasn't.

3. If two events are mutually exclusive are they also exhaustive? If so explain, if not give a counter example.
Not necessarily. For example, getting a 1 and getting a 3 on a dice are mutually exclusive, but not exhaustive because you could also get a 2,4,5,6.

4. If two events are exhaustive, are they also mutually exclusive? If so explain, if not give a counter example.
Not necessarily. For example in a family of 3 the events "getting least one son" and "getting at least one daughter" are exhaustive events because no matter how many of each gender you get you either have at least one son or at least one daughter. They are not mutually exclusive since, for example, having a one son and two daughters satisfies both events.

5. Use the laws above to answer the following questions

   (a) In a family of 4 children, what is the probability of getting only girls?
   The probability that any given child is a girl is $\frac{1}{2}$, so the probability that the first, second, third and forth is a girl is $(\frac{1}{2})(\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) = \frac{1}{16}$.

   (b) In a family of 4 children, what is the probability of getting at least one boy?
   If A is the event of getting at least one boy then A' is the event of not getting at least one boy – that is getting all girls. $P(A) = 1 - P(A') = 1 - \frac{1}{16} = \frac{15}{16}$.

   (c) Toss a coin and draw a card, what is the probability of getting a head and an ace?
   $P(\text{Head}) = \frac{1}{2}$ and $P(\text{Ace}) = \frac{1}{13}$ so since they are independent events $P(\text{Head and Ace}) = (\frac{1}{2})(\frac{1}{13}) = \frac{1}{26}$.

   (d) What is the probability of getting a head or an ace? (Hint: You will need to use the more general law of addition).
   Since these ar not mutually exclusive events we use the general law $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = \frac{1}{2} + \frac{1}{13} - \frac{1}{26} = \frac{13+2-1}{26} = \frac{14}{26} = \frac{7}{13}$.

   (e) Suppose that if a father is blue eyed and a mother is brown eyed then the probability of having a child with blue eyes is $\frac{1}{4}$. If such parents have four children what is the probability of getting at least one blue eyed child.
   Let A be the event of getting at least one blue eyed child. Then A' is the probability of getting no blue eyed child. $P(A') = (\frac{1}{4})^4 = \frac{1}{256}$ so $P(A) = 1 - P(A') = 1 - \frac{1}{256} = \frac{255}{256}$.

6. Consider the following game modeling the interaction between two individuals. In this type of interaction there is a task to be completed (eg building a nest, or dam), from which they both get benefit b. If they both work to complete the task they each incur a cost $\frac{1}{2}c$, but if only one does the work that individual incurs cost c and the other incurs no cost. There are two strategies: Work and Shirk.

   (a) Write down the payoff matrix for this interaction game.

	W	S
W	$b - \frac{1}{2}c$	$b - c$
S	b	0

   (b) Can Work be an ESS? If so, under what conditions?
   For work cannot ESS since $b - \frac{c}{2} < b$

   (c) Can Shirk be an ESS? If so, under what conditions?
   Shirk can be an ESS whenever $0 > b - c$ which means $b < c$.

   (d) Under the condition where neither Work nor Shirk is an ESS, there is an equilibrium mixture of x Workers and $1 - x$ Shirkers for which a Worker and Shirker are equally fit. Find this particular value of x.
   The fitness of the Workers is

$$f_W = x(b - \tfrac{1}{2}c) + (1 - x)(b - c) = \tfrac{1}{2}x\,c + b - c\,.$$

   The fitness of Shirkers is $f_S = b\,x$. So $f_S = f_W$ implies

$$b\,x = \tfrac{1}{2}x\,c + b - c \;\Rightarrow\; (b - \tfrac{1}{2}c)x = b - c \;\Rightarrow\; x = \frac{b - c}{b - \frac{1}{2}c} = \frac{2(b - c)}{2b - c}$$

   (e) Find out the average fitness of the population at the equilibrium and compare that to the average fitness of a population of only Workers.
   At the equilibrium the fitness of the players is $b\,x$ which gives

$$\frac{b(b - c)}{b - \frac{1}{2}c}$$

   The fitness of population of only Workers is $b - \frac{1}{2}c$. Thus a population of pure Workers is fitter than the equilibrium mixture when

$$b - \tfrac{1}{2}c > \frac{b(b - c)}{b - \frac{1}{2}c} \;\Rightarrow\; (b - \tfrac{1}{2}c)^2 > b(b - c) \;\Rightarrow\; b^2 - bc + (\tfrac{1}{2}c)^2 > b(b - c) \;\Rightarrow\; (\tfrac{1}{2}c)^2 > 0\,,$$

   which is always true.

7. Consider all possible 2×2 symmetric games. There are 24 ways to arrange the numbers 1,2,3 and 4 as payoffs in the game. If we agree to make the diagonal entry at AA larger than the diagonal entry at BB then there are only 12 (why is it reasonable to do this?). Write out all 12 games and classify them into the cases where A is the only ESS, B is the only ESS, A and B are both ESS, and none are ESS. Now list those games where the ESS leads to a payoff that is suboptimal, in the sense that the population would have a higher payoff if all chose another strategy. Which of these cases is the prisoner's dilemma game. Which is similar to the hawk-dove game?

Here is the list:

$\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$	A and B ESS	$\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$	A and B ESS
$\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$	A and B ESS	$\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}$	Only A ESS
$\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$	Only A ESS	$\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$	Only A ESS
$\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$	Only B ESS	$\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$	Neither ESS
$\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$	Only A ESS	$\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$	Only A ESS
$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$	Neither is ESS	$\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$	Neither is ESS

The only two that have suboptimal equilibria are $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$ (which leads to an average payoff of 2, when a population of A players gets 3) and $\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$ (which leads to a mixed equilibrium with average fitness 2.5 when a population of A players gets 3). The first of these games is the prisoner's dilemma and the second is chicken. The last two are also interesting games because if one player would agree to play A whenever the other played B then one would get 4 and one 3 . If they took turns then their average fitness would be 3.5 which is better than the 2.5 average fitness of the mixed equilibrium. This game is sometimes called the coordination game. Some coordination strategies would be something like: play A if smaller and B if larger. Then when two meet, the larger gets 4 and the smaller gets 3.