2. all blocking coalitions containing the first voter.

3. all voters that have veto power.

4. all dummy voters.

(a) [51:52, 48]
(b) [2:1, 1, 1]
(c) [3:2, 2, 1]
(d) [8:5, 4, 3]
(e) [51:45, 43, 8, 4]
(f) [51:28, 27, 26, 19]
(g) [16:10, 10, 10, 1]
(h) [21:10, 10, 10, 10, 1]

4. List all of the winning coalitions of a committee of four members, A, B, C, and D, with voting system [51:30, 25, 24, 21].

5. How would the list in Exercise 4 change if the quota were increased to

(a) 52?
(b) 55?
(c) 58?

6. Voter A in Exercise 4 would like to have veto power. How much should the quota be increased to give her, and no one else, veto power?

The Banzhaf Power Index

7. (a) List the 16 possible combinations of how four voters, A, B, C, and D, can vote either yes (Y) or no (N) on an issue.
   (b) List the 16 subsets of the set {A, B, C, D}.
   ◆ (c) How do the lists in parts (a) and (b) correspond to each other?
   (d) In how many of the combinations in part (a) is the vote

   (i) 4 Y to 0 N?
   (ii) 3 Y to 1 N?
   (iii) 2 Y to 2 N?

8. Calculate the number of extra votes for each of the winning coalitions found in Exercise 4. Identify the winning coalitions in which

(a) A is a critical voter.
(b) B is a critical voter.

9. Calculate the Banzhaf index for the voting system in Exercise 4.

10. The system in Exercise 4 is modified by increasing the quota to

(a) 52
(b) 55
(c) 58
(d) 73
(e) 76
(f) 79
(g) 82

Calculate the Banzhaf index in each case. (*Hint:* Increasing the quota will reduce the number of extra votes in each of the original coalitions. When the number of extra votes becomes negative, the coalition is losing and you can cross it off the list. As the number of extra votes decreases, a member of a coalition who was originally not a critical voter will become a critical voter.)

11. Calculate the Banzhaf index for each of the weighted voting systems in Exercise 3.

12. Calculate the following:

(a) C_3^6
(b) C_{100}^{50}
(c) C_3^{10}
(d) C_7^{10}

13. Calculate the following:

(a) C_4^6
(b) C_2^{100}
(c) C_{98}^{100}
(d) C_5^{10}

14. The Board of Supervisors of Nassau County, New York, is a historically important example of a weighted voting system (see Spotlight 12.3). Before it was declared unconstitutional by a federal district court in 1993, the weighted voting system of the

16. A committee has a chairperson and six ordinary members. It uses majority rule, except that the chairperson is only allowed to vote when it is necessary to break a tie. Give an equivalent weighted voting system for the committee. What would happen if one of the members is absent?

◆ 17. A five-member committee has the following voting system. The chairperson can pass or block any motion that she supports or opposes, provided that at least one other member is on her side. Show that this voting system is equivalent to the weighted voting system $[4:3, 1, 1, 1, 1]$.

18. Calculate the Banzhaf index for the weighted voting system in Exercise 17.

19. Which of the following voting systems are equivalent to weighted voting systems? Find the weights and quota for those that are.

(a) A committee of three faculty and the dean. To pass a measure, at least two faculty members and the dean must vote yes.
(b) A committee of three faculty, the dean, and the provost. To pass a measure, two faculty, the dean, and the provost must vote yes.
(c) A four-member faculty committee and a three-member administration committee vote separately on each issue. The measure passes if it receives the support of a majority of each of the committees.

20. Calculate the Banzhaf index of each of the voting systems in Exercise 19.

◆ 21. How many *distinct* (nonequivalent) voting systems with four voters can you find? Systems that have dummies don't count. The challenge is to find all nine.

The Shapley–Shubik Power Index

22. For the voting system in Exercise 4, list all permutations of the voters in which

(a) A is the pivotal voter.
(b) B is the pivotal voter.

23. Calculate the Shapley–Shubik index for the system in Exercise 4.

24. Calculate the Shapley–Shubik index for the weighted voting system in Exercise 17.

25. Calculate the Shapley–Shubik index of each of the voting systems in Exercise 19.

Systems with Large Numbers of Voters

26. A corporation has 120 shares of stock outstanding. There are 100 shareholders who own one share each, and one shareholder who owns 20 shares. To pass an issue, owners representing 61 shares must vote yes. Determine the Shapley–Shubik index of each shareholder.

■ 27. Estimate the power of the shareholders of the corporation in Exercise 26, as measured by the Banzhaf index.

Miscellaneous Problems

◆ 28. The vice president of the United States is allowed to break ties in the U.S. Senate. How does his or her Banzhaf power index compare with that of an individual senator?

29. Determine the Shapley–Shubik power index for the four-person voting system described in Exercise 15.

30. A corporation has four shareholders and a total of 100 shares. The quota for passing a measure is the votes of shareholders owning 51 or more shares. The number of shares owned are as follows:

A 48 shares
B 23 shares
C 22 shares
D 7 shares

All transactions must be in whole numbers of shares; sales of fractional shares are not permitted.

(a) List the winning coalitions and compute the number of extra votes for each. Make a separate list of the losing coalitions and compute the number of votes that would be

needed to make the coalition winning.

(b)   How many shares can A sell to B without causing any of the winning coalitions listed in part (a) to lose, or any of the losing coalitions in part (a) to win?

(c)   How many shares can A sell to D without changing the sets of winning or losing coalitions?

(d)   E, who now owns none of the stock, would like to buy some. How many shares can A sell to E without changing the winning coalitions? Note that since E is now a dummy, he must remain a dummy after the trade.

(e)   How many shares can D sell, without changing the set of winning coalitions, to A, B, C, or E? Again, it is conceivable that D would be able to sell more to one stockholder than to another.

(f)   How many shares can D sell to A, B, C, or E without becoming a dummy?

(g)   How many shares can B sell to C without changing the set of winning coalitions?

31.   Which of the following voting systems is equivalent to the voting system in use by the corporation in Exercise 30?

(a) $[3:1, 1, 1, 1]$
(b) $[3:2, 1, 1, 1]$
(c) $[5:3, 1, 1, 1]$
(d) $[5:3, 2, 1, 1]$
(e) $[5:3, 2, 2, 2]$

32.   Determine the Banzhaf and Shapley–Shubik power indices for the corporation in Exercise 30.

33.   (a)   Show that $C_3^5 + C_4^5 = C_4^6$.
(b)   Show that $C_5^9 + C_6^9 = C_6^{10}$.
◆ (c)   Explain why the following combinatorial identity is true.

$$C_{k-1}^n + C_k^n = C_k^{n+1}$$

(*Hint:* Consider k-member coalitions in an n + 1– member committee. Count those that have the chairperson as a member, and those that do not include the chairperson.)

34.   A nine-member committee has a chairperson and eight ordinary members. A motion can pass if and only if it has the support of the chairperson and at least two other members, or if it has the support of all eight ordinary members.

(a)   Find an equivalent weighted voting system.
(b)   Determine the Banzhaf power index.
(c)   Determine the Shapley–Shubik power index.

35.   Consider the 2m-person voting system in which each participant has one vote and a simple majority wins. In the notation for weighted voting systems, this system can be expressed as

$$[m + 1:1, \ldots, 1]$$

Assume that all voting combinations are equally likely. What is the probability that a voter will be a critical voter, when m = 1, 2, 3, 4, 5, 6, or 7?

36.   The New York City Board of Estimate consists of the mayor, the comptroller, the city council president, and the presidents of each of the five boroughs. It employed a voting system in which the city officials each had two votes and borough presidents each had one; the quota to pass a measure was six. This voting system was declared unconstitutional by the U.S. Supreme Court in 1989 (*Morris* v. *Board of Estimate*).

(a)   Describe the minimal winning coalitions.
▨ (b)   Determine the Banzhaf power index.

37.   Here is a proposed weighted voting system for the New York City Board of Estimate that is based on the populations of the boroughs (see Exercise 36):

$$[71:35, 35, 35, 11.3, 7.3, 9.6, 6.0, 1.8]$$

Find a simpler system of weights that yields an equivalent voting system.

38.   The United Nations Security Council has five permanent members: China, France, Russia, the United Kingdom, and the United States, and 10 other members that serve 2-year terms. To resolve a