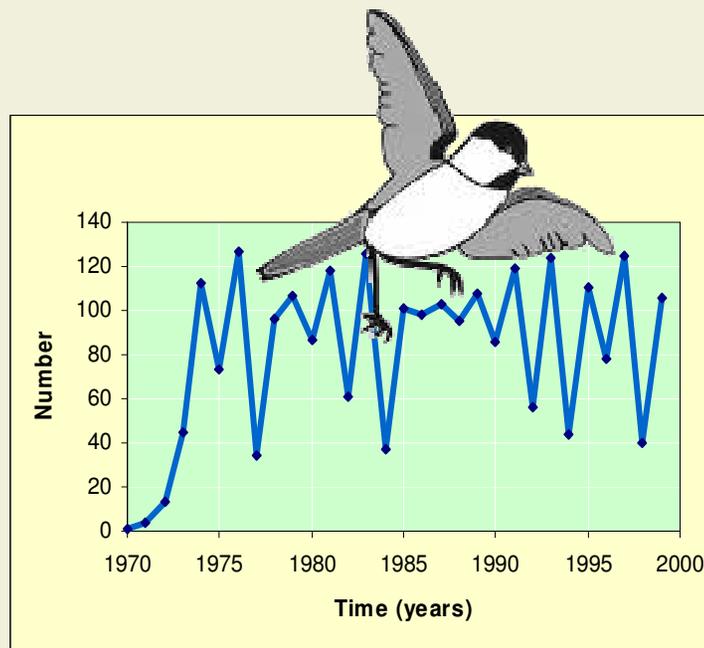


# Chaos in Population Dynamics

## Understanding chaos in the logistic model



*How does chaos emerge in deterministic models of population growth? What are the quantitative tools we need to understand and represent chaotic behavior?*

Core Quantitative concepts and skills  
Modeling with difference equations

Supporting Quantitative concepts and skills  
Visual Display of Data – XY scatter plots  
Iteration  
Exponential and logistic growth  
Creating cobweb diagrams  
Curve fitting and trend lines

Prepared for SSAC by  
\*David McAvery – The Evergreen State College\*

# Problem

The logistic model is used to describe the growth of a population in an environment with limited resources. When the population grows continuously (that is, when the births and deaths do not occur at fixed intervals) the logistic model always leads to smooth predictable growth of the population to a limiting value called the carrying capacity, which is the maximum value the environment can support. However, when the population grows discretely (for example, a population that reproduces at fixed intervals – such as once a year in spring) then the dynamics can be more complicated. The model can lead to a population that reaches a simple equilibrium, has periodic oscillations or exhibits chaotic behavior, depending on the population growth rate for the discrete time interval. In this module we will assume a time interval of one year and will explore the dynamics of the discrete logistic model with different growth rates.

The problem:

- What values of the growth rate in the logistic model lead to a simple equilibrium and how does the model approach this equilibrium?
- What values of the growth rate lead to periodic behavior and what type of periodicity can result?
- What values of the growth rate lead to chaotic population dynamics in the logistic model?
- What methods can we use to visualize and quantify the approach to chaos?

# The Logistic Model

Our first task is to create a spread sheet implementing the recursion relation for the logistic model:  $P_n = P_{n-1} + aP_{n-1}(1 - P_{n-1}/K)$ , where  $P_n$  is the population number,  $a$  is the intrinsic growth rate and  $K$  is the carrying capacity. In this example we will start with an initial population of 1, a growth rate of 0.1 per year and a carrying capacity of 100.

|    | B     | C     |
|----|-------|-------|
| 2  |       |       |
| 3  | $P_0$ | 1     |
| 4  | $a$   | 0.1   |
| 5  | $K$   | 100   |
| 6  |       |       |
| 7  | $n$   | $P_n$ |
| 8  | 0     | 1     |
| 9  | 1     | 1.10  |
| 10 | 2     | 1.21  |
| 11 | 3     | 1.33  |
| 12 | 4     | 1.46  |
| 13 | 5     | 1.60  |
| 14 | 6     | 1.76  |
| 15 | 7     | 1.93  |
| 16 | 8     | 2.12  |
| 17 | 9     | 2.33  |
| 18 | 10    | 2.56  |
| 19 | 11    | 2.81  |
| 20 | 12    | 3.08  |
| 21 | 13    | 3.38  |
| 22 | 14    | 3.70  |

 = cell with a number in it  
 = cell with a formula in it

Create a worksheet like this one, giving an recursive solution to the logistic model.

Hint: Enter the equation for the logistic recursion equation here

Hint: Copy the formula down for 100 cells.

Make sure you use absolute references for the parameter values.

Hint: Need help with entering formulas?

Here is an example: If you have a number in Cell B2 and you want to calculate the square of it in Cell C2 then in C2 you type =B2^2.

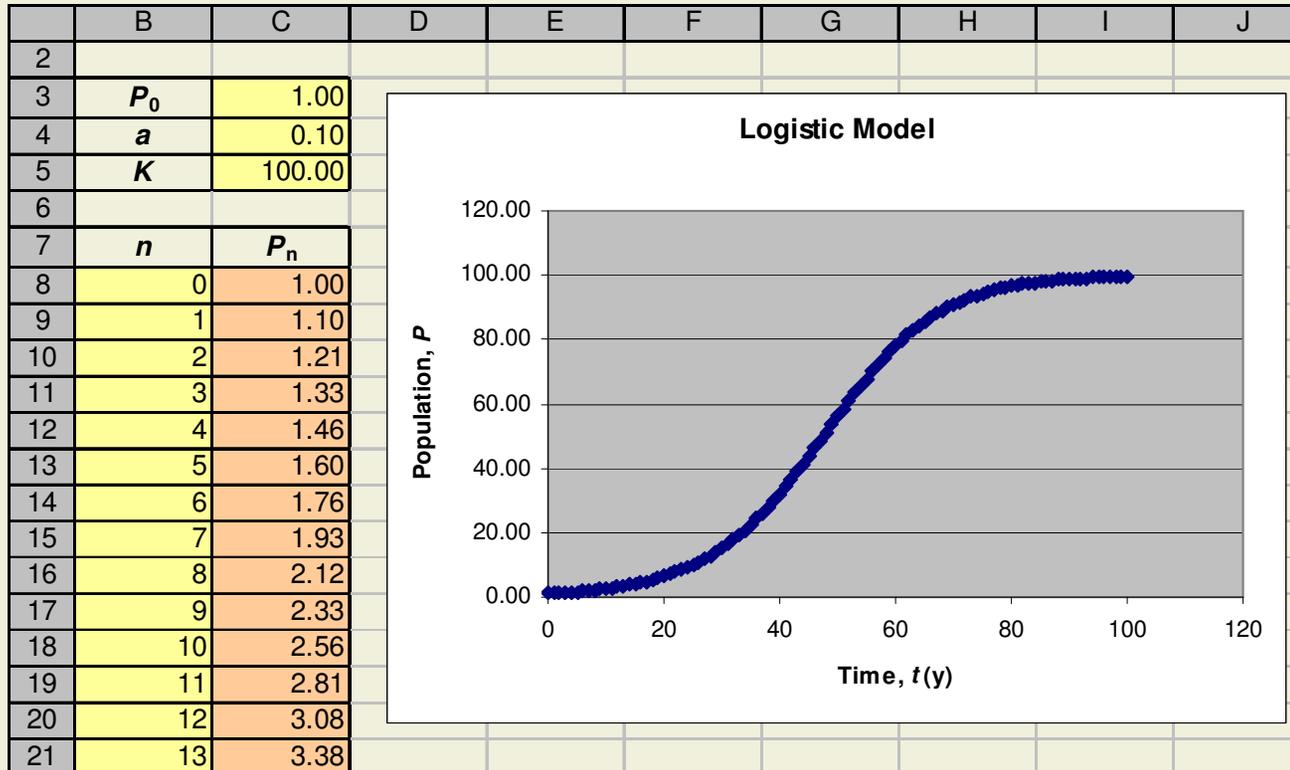
|   | B | C       |
|---|---|---------|
| 2 | 2 | 4 =B2^2 |

To use absolute references type =\$B\$2^2 instead.

|   | B | C           |
|---|---|-------------|
| 2 | 2 | 4 =\$B\$2^2 |

# Graphing the Solution

Create a plot of the population of the function of time as predicted by the logistic model.



Use an XY scatter plot with points connected. Label the axes and format the graph correctly.

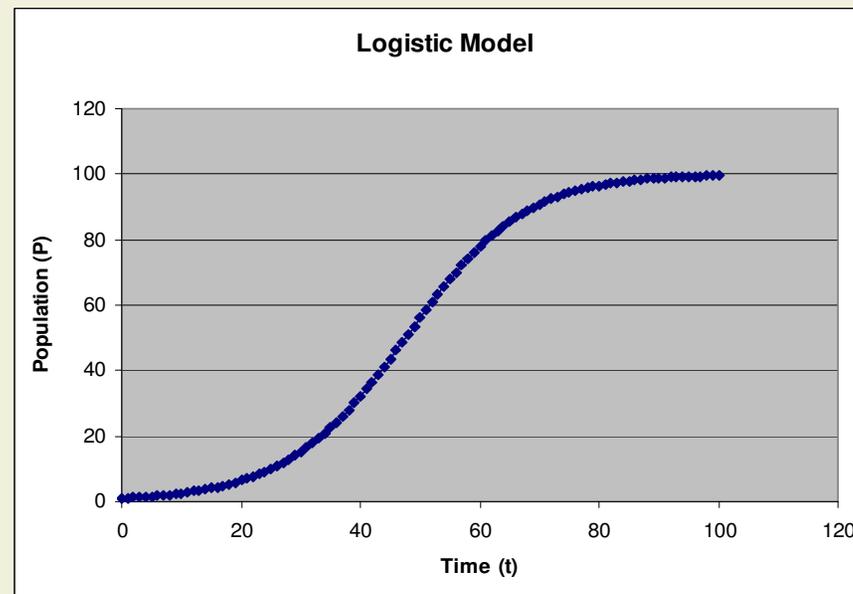
## Exploring Parameter Space

Now we are ready to explore how the solution to the logistic model depends on the three parameters  $P_0$ ,  $a$  and  $K$ .

Answer the following questions in a textbox in a new sheet named **assignments**.

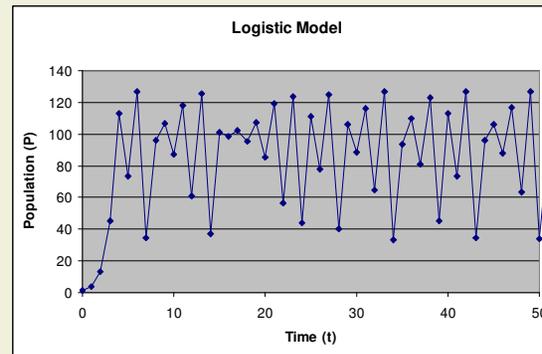
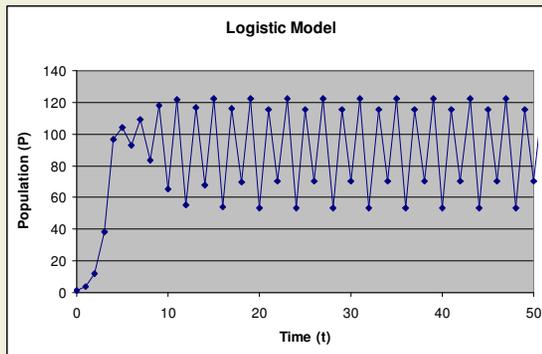
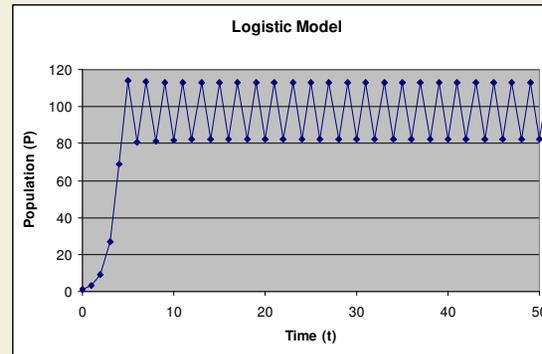
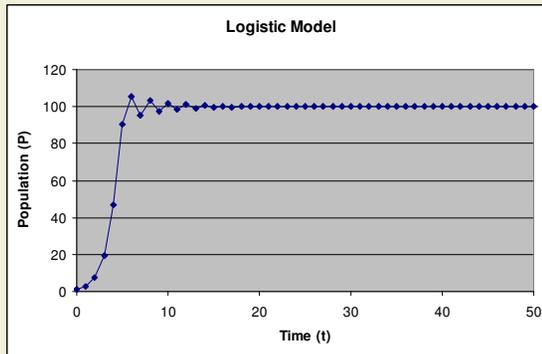
- Vary the initial population  $P_0$  between 0 and 200 while keeping  $a=0.1$  and  $K=100$ . Describe the solution in each case. What do the solutions have in common and how do they differ?
- Vary the carrying capacity  $K$  between 0 and 200 while keeping  $a=0.1$  and  $P_0=1$ . Describe the solution in each case. What do the solutions have in common and how do they differ?
- Vary the growth rate  $a$ , between 0 and 1 while keeping  $K=100$  and  $P_0=1$ . Describe the nature of the solution in each case. What do the solutions have in common and how do they differ?

|   | B     | C   |
|---|-------|-----|
| 2 |       |     |
| 3 | $P_0$ | 1   |
| 4 | $a$   | 0.1 |
| 5 | $K$   | 100 |



# The Route to Chaos

When the growth rate,  $a$ , varies between 0 and 1, the logistic model predicts a smooth approach to equilibrium at a population equal to the carrying capacity. However, if you consider values of the growth rate larger than 1 the model behaves in unexpected ways. For some values of  $a$  the solution overshoots the carrying capacity before approaching equilibrium, for some values it oscillates between two or more equilibria (called 2-cycles, 4-cycles, *etc*), and for other values the solution is chaotic.



Answer the following question in your **assignments** sheet.

2. Find the range of  $a$  for which the model exhibits:

- (a) oscillating approach to equilibrium.
- (b) 2-cycles.
- (c) 4-cycles.
- (d) chaos.

## Quantifying Chaos

The word *chaos* is often used in imprecise ways. In this module, a chaotic system means one that exhibits sensitive dependence on initial conditions. If a model predicts two vastly different results for two populations which initially differ by only a small amount then the model is said to be chaotic. More precisely, if we define the absolute value of the difference between two populations as  $\delta$ , then if the ratio of  $\delta$  to the initial absolute difference  $\delta_0$  diverges exponentially the system is chaotic.

|    | B          | C         | D        | E                 |
|----|------------|-----------|----------|-------------------|
| 2  |            |           |          |                   |
| 3  | $P_0$      | 50        |          |                   |
| 4  | $a$        | 2.75      |          |                   |
| 5  | $K$        | 100       |          |                   |
| 6  | $\delta_0$ | 0.00001   |          |                   |
| 7  | $n$        | $P_n$     | $P_n$    | $\delta/\delta_0$ |
| 8  | 0          | 50        | 50.00001 | 1                 |
| 9  | 1          | 118.75    | 118.75   | 1                 |
| 10 | 2          | 57.519531 | 57.5195  | 2.78125           |
| 11 | 3          | 124.71459 | 124.7146 | 1.630999          |
| 12 | 4          | 39.952169 | 39.95222 | 5.071267          |
| 13 | 5          | 105.9258  | 105.9259 | 7.873798          |
| 14 | 6          | 88.664187 | 88.66402 | 16.34538          |
| 15 | 7          | 116.3039  | 116.3041 | 18.4135           |
| 16 | 8          | 64.158191 | 64.1577  | 48.73536          |
| 17 | 9          | 127.3957  | 127.3956 | 10.78576          |
| 18 | 10         | 31.418119 | 31.41847 | 35.12665          |
| 19 | 11         | 90.672745 | 90.67345 | 71.02587          |
| 20 | 12         | 113.93026 | 113.9294 | 87.86046          |
| 21 | 13         | 70.285611 | 70.28782 | 221.0692          |

- Make a copy of your first spreadsheet to a new sheet. Then create an entry for the initial difference in populations  $\delta_0$ .

- Add a new population column, whose initial value is  $P_0 + \delta_0$ . You may also want to format the columns so that they show enough decimal places to illustrate the small difference in the populations.

- Finally make a column where you calculate  $\delta/\delta_0$ . This column only needs to extend about 30 time steps. You might observe that this value jumps around a bit, However you should see that it has a tendency to grow when  $a$  is in regions where the population dynamics is chaotic.

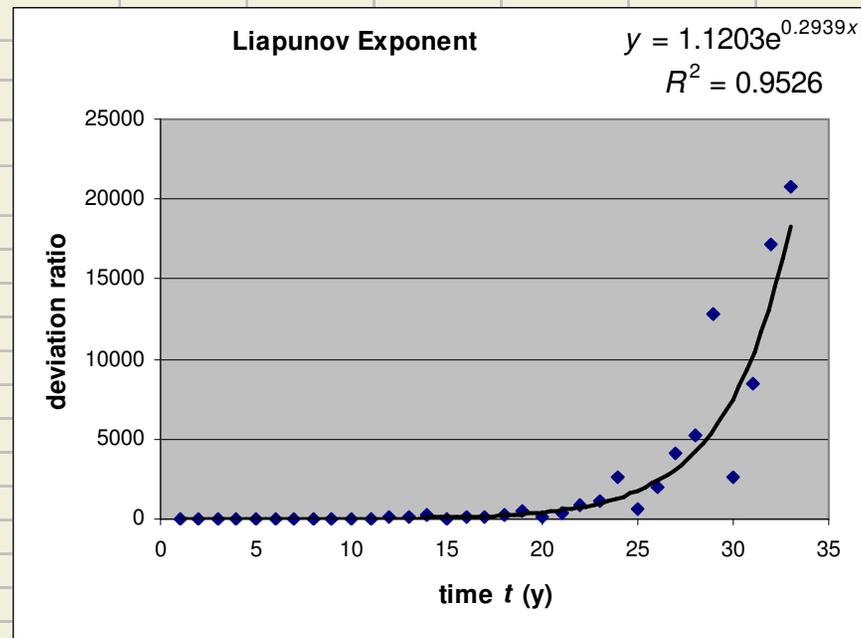
# The Liapunov Exponent

To determine if the ratio  $\delta/\delta_0$  grows exponentially we plot  $\delta/\delta_0$  vs time and then fit an exponential curve  $e^{\lambda t}$  to the data. The exponent  $\lambda$  is called the Liapunov exponent, which is a quantitative measure of chaos. A value of  $\lambda$  less than or equal to zero implies that the system is not chaotic.

Plot  $\delta/\delta_0$  vs  $t$ . Then add an exponential trend line. Make sure you choose the option to include the equation and  $R^2$  value of trend line on the graph. Now vary the value of the growth rate  $a$  and determine the value of the Liapunov exponent  $\lambda$  for different values of  $a$ .

3. Find the range of values for  $a$  where the Liapunov is positive (indicating chaos).

|    | B          | C         | D        | E                 | G | H | I | J | K | L | M |
|----|------------|-----------|----------|-------------------|---|---|---|---|---|---|---|
| 2  |            |           |          |                   |   |   |   |   |   |   |   |
| 3  | $P_0$      | 50        |          |                   |   |   |   |   |   |   |   |
| 4  | $a$        | 2.75      |          |                   |   |   |   |   |   |   |   |
| 5  | $K$        | 100       |          |                   |   |   |   |   |   |   |   |
| 6  | $\delta_0$ | 0.00001   |          |                   |   |   |   |   |   |   |   |
| 7  | $n$        | $P_n$     | $P_n$    | $\delta/\delta_0$ |   |   |   |   |   |   |   |
| 8  | 0          | 50        | 50.00001 | 1                 |   |   |   |   |   |   |   |
| 9  | 1          | 118.75    | 118.75   | 1                 |   |   |   |   |   |   |   |
| 10 | 2          | 57.519531 | 57.5195  | 2.78125           |   |   |   |   |   |   |   |
| 11 | 3          | 124.71459 | 124.7146 | 1.630999          |   |   |   |   |   |   |   |
| 12 | 4          | 39.952169 | 39.95222 | 5.071267          |   |   |   |   |   |   |   |
| 13 | 5          | 105.9258  | 105.9259 | 7.873798          |   |   |   |   |   |   |   |
| 14 | 6          | 88.664187 | 88.66402 | 16.34538          |   |   |   |   |   |   |   |
| 15 | 7          | 116.3039  | 116.3041 | 18.4135           |   |   |   |   |   |   |   |
| 16 | 8          | 64.158191 | 64.1577  | 48.73536          |   |   |   |   |   |   |   |
| 17 | 9          | 127.3957  | 127.3956 | 10.78576          |   |   |   |   |   |   |   |
| 18 | 10         | 31.418119 | 31.41847 | 35.12665          |   |   |   |   |   |   |   |
| 19 | 11         | 90.672745 | 90.67345 | 71.02587          |   |   |   |   |   |   |   |
| 20 | 12         | 113.93026 | 113.9294 | 87.86046          |   |   |   |   |   |   |   |



Hint: Need help with trend lines? Right click on a data point in your graph. Select **add trend line** from the menu. Choose the exponential fit and then under options check the boxes for **display equation on chart** and **display R-squared value on chart**.

## Illustrating Dynamics of the Logistic Model with Cobweb Plots

As you discovered earlier, the logistic model exhibits a range of different behaviors depending on the values of the growth rate  $a$ . The cobweb diagram looks qualitatively different in each case, and illustrate why the model model changes its behavior at particular values of the growth rate.

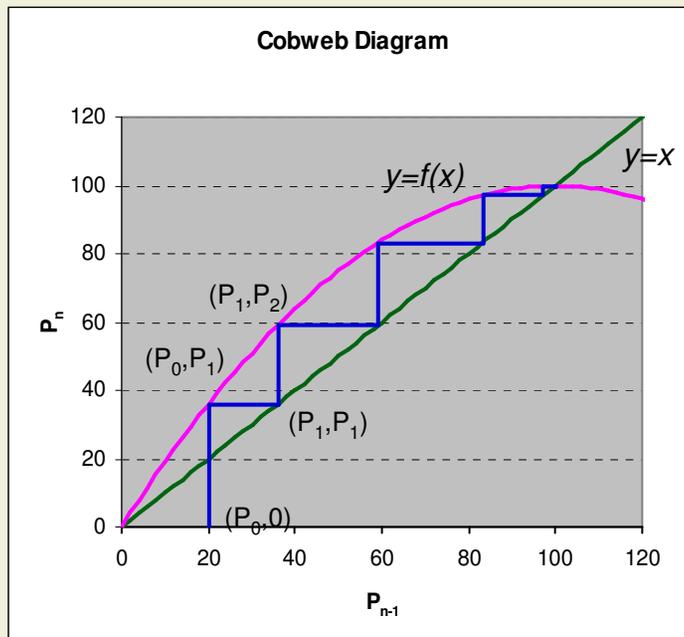
Answer the following question in your **assignments** sheet.

4. Choose values of  $a$  corresponding to each of the cases below and create cobweb plots. describe the cobweb plots, qualitatively noting differences between the cases. Find values of  $a$  corresponding to transitions between two types of behavior and then note, if possible, what feature of the plot is changing at that point. Pay particular attention to how the cobweb plot behaves near the intersection between  $y=f(x)$  and  $y=x$ , and the sign of the slope of  $y=f(x)$  there.

- (a) smooth approach to equilibrium
- (b) oscillating approach to equilibrium
- (c) 2-cycles
- (d) 4-cycles
- (e) chaos

# Creating a Cobweb Plot – Part 1

In order to understand how the different types of behavior in the logistic model come about it is useful to create a cobweb plot. A cobweb plot is a graph of  $P_n$  vs  $P_{n-1}$  and is used to illustrate how the sequence of values is generated by the difference equation,  $P_n = f(P_{n-1})$ . For the logistic model it follows from Slide 4 that  $f(x) = x + a x (1 - x/K)$ . In the cobweb plot we start with the initial input value  $x = P_0$ , move up to the curve  $y = f(x)$  to get the value  $y = P_1$  and return to the line  $y = x$  to find the next input value  $x = P_1$ . We then repeat the process. The sequence of points to plot is  $(P_0, 0)$ ,  $(P_0, P_1)$ ,  $(P_1, P_1)$ ,  $(P_1, P_2)$  ... See the graph below.



Copy your spreadsheet to a new one and then modified it to create the rows of points as suggested below.

|    | B     | C         | D     |
|----|-------|-----------|-------|
| 2  |       |           |       |
| 3  | $P_0$ | 20        |       |
| 4  | $a$   | 1         |       |
| 5  | $K$   | 100       |       |
| 6  |       |           |       |
| 7  | $n$   | $P_{n-1}$ | $P_n$ |
| 8  |       | 20.00     | 0.00  |
| 9  | 0     | 20.00     | 36.00 |
| 10 |       | 36.00     | 36.00 |
| 11 | 1     | 36.00     | 59.04 |
| 12 |       | 59.04     | 59.04 |
| 13 | 2     | 59.04     | 83.22 |
| 14 |       | 83.22     | 83.22 |
| 15 | 3     | 83.22     | 97.19 |
| 16 |       | 97.19     | 97.19 |
| 17 | 4     | 97.19     | 99.92 |
| 18 |       | 99.92     | 99.92 |

The initial point is  $(P_0, 0)$

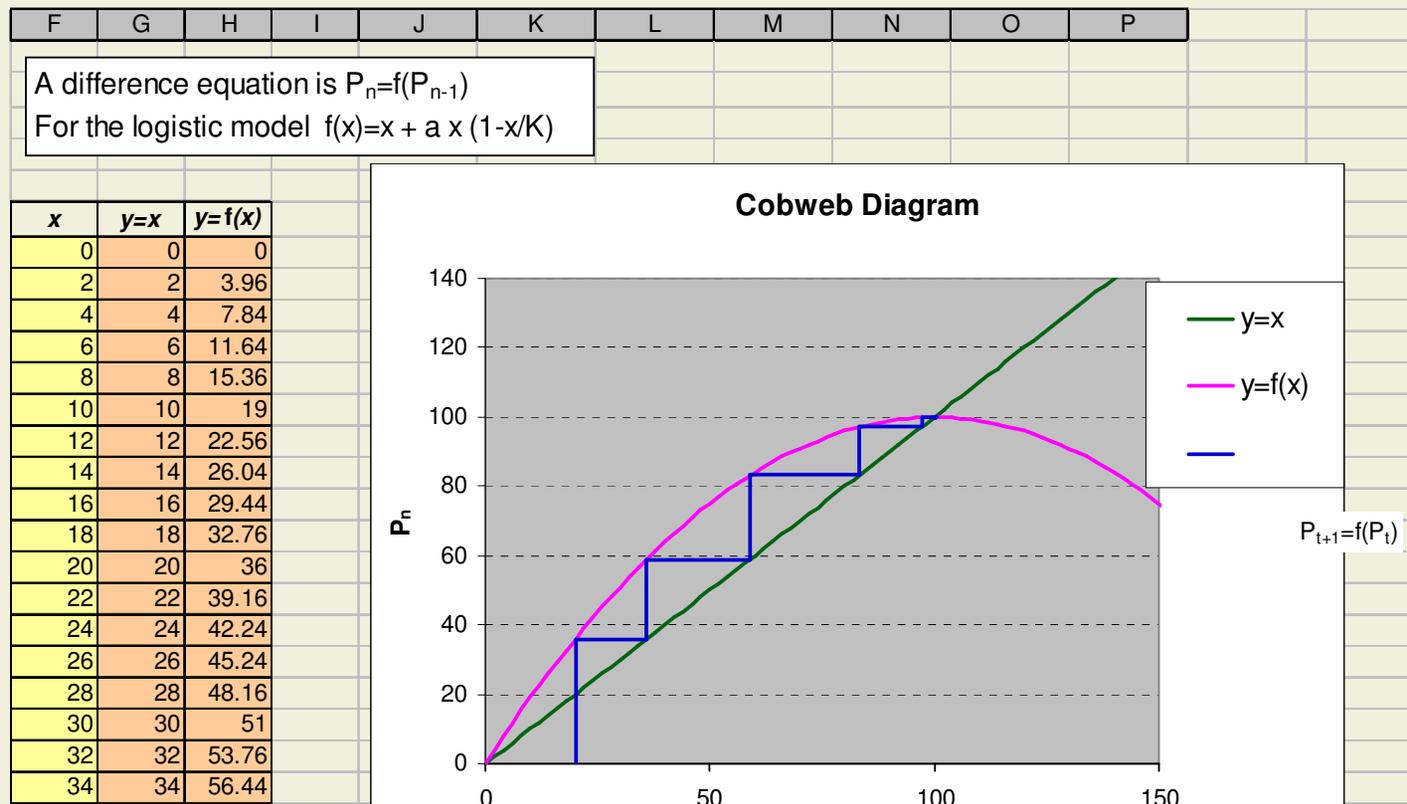
Insert a row for the point which brings the cobweb plot back to the  $y=x$  line.

Now copy the second and third rows down.

## Creating a Cobweb Plot – Part 2

In order to create the full cobweb plot we need to plot the line  $y=x$ , the curve  $y= f(x)=x + a x (1-x/K)$  and the sequence of points from the previous slide.

Add new columns in your spreadsheet that will allow you to plot  $y=x$  and  $y= f(x)=x + a x (1-x/K)$ . You need to first include a column of  $x$  values. These should range between 0 and 200 (or twice the carrying capacity). When you have added your columns, plot  $y=x$  and  $y= f(x)$  and  $P_n$  vs  $P_{n-1}$  on the same graph. Use XY scatter plot with connected points. Use different colors for each curve.



## Assignment

Write your answers to the following questions in your “Assignment” spreadsheet. When you are finished the lab submit your spreadsheet to the program Moodle Page. Due Tuesday, January 25th at 1:00 pm.

1. Answer the questions about the parameters of the logistic model on Slide 5.
2. Answer the questions about the range of values of the growth rate  $a$  corresponding to different types of behavior of the logistic model on Slide 6.
3. Answer the question about the Liapunov exponent on Slide 8.
4. Answer the questions about the cobweb diagrams on Slide 9.
5. Is the fact that the logistic model yields chaotic solutions for high values of the growth rate reasonable on biological grounds? Explain your answer.