Allman 3.1 Ex 2,6,7

3.1.2. a. The peaks on the graph of $P_t = \cos t$ lead those on the graph of $Q_t = \sin t$ (by a time interval of $\pi/2$), similar to those of the prey and predator.
b. The plotted points lie on a circle centered at the origin, starting on the x-axis when $t = 0$ and proceeding counterclockwise around the circle.
c. Because the oscillations in the first figure get smaller as time increase (i.e., they are damped), the spiral in the second figure goes inward.

3.1.6. With the other parameters as in the text, $r = 2.1$ results in what appears to be a stable equilibrium, despite the fact that in the logistic model it leads to a 2-cycle. This illustrates that a predator-prey interaction can have a stabilizing effect on otherwise complex dynamics. A real-world example of this involves deer populations and hunters. (Can you find parameter values for which the logistic model is chaotic, yet the predator-prey model has an apparently stable equilibrium?)

3.1.7. The parameter w represents the size of the prey population that can be protected in the refuge. If $P > w$, then $P - w$ is the part of the prey population not in the refuge, which is therefore subject to the predator-prey interaction. The given interaction terms thus describe the predation appropriately. If $P < w$, however, the terms are not correct. Replacing the $P - w$ with $\max(0, P - w)$ would be better.

Allman 3.2 Ex 1,3,7

3.2.1. If $u/v = 1$, the vertical line of the Q-nullcline joins the sloping line of the P-nullcline on the P-axis at $P = 1$. Then the only equilibria are $(0,0)$ and $(1,0)$. If $u/v > 1$, the vertical line lies even further to the right, and intersects the sloping line below the P-axis. The resulting equilibrium has $Q^* < 0$, so $(0,0)$ and $(1,0)$ are the only two biologically meaningful equilibria..

3.2.3. The nullclines are described in problem 3.2.1. For both $u/v = 1$ and $u/v > 1$ the region under the sloping line should have arrows pointing down and to the right. The region to the right of the vertical line should have arrows pointing up and to the left. The remaining region should have arrows pointing down and to the left. MATLAB experiments confirm this.

3.2.7. a. Both predator and prey are follow the logistic model in the absence of the other, but the extra terms mean the predator benefits and prey is harmed from the predator-prey interaction.

b. The P-nullcline is as in Figure 3.4. The Q-nullcline is the P-axis ($Q = 0$) together with the line $P = (u/v)(Q - 1)$ which goes through $(0, 1)$ and slopes upward. If $r/s > 1$, the two sloping lines of the nullclines intersect, and produce four regions. If $r/s \leq 1$, there are only three regions. Below $P = (u/v)(Q - 1)$ arrows point up; above it they point down. Above the line $Q = (r/s)(1 - P)$ arrows point to the left; below it they point to the right.

c. Equilibria are at $(0, 0)$, $(1, 0)$ and $((r - s)u/(ru + vs), (u + v)r/(ru + vs))$. The third equilibrium is only biologically meaningful if $r/s > 1$.

d. For $r/s > 1$ you might expect orbits to move counterclockwise around the third equilibrium, provided they begin close enough to it. Whether they spiral inward or outward is not yet clear.

_Allman Ch 1.2, Ex 2,5,7,8,9,11

1.2.2. ΔP will be positive for any value of $P < 10$ and ΔP will be negative for any value of $P > 10$. Assuming $P > 0$ so that the model has a meaningful biological interpretation, we see that a population increases in size if it is smaller than the carrying capacity $K = 10$ of the environment, and decreases when it is larger than the environment's carrying capacity.

1.2.5. a. $\Delta P = 2P(1 - P/10)$; $\Delta P = 2P - .2P^2$; $\Delta P = .2P(10 - P)$; $P_{t+1} = 3P_t - .2P_t^2$

b. $\Delta P = 1.5P(1 - P/(7.5))$; $\Delta P = 1.5P - .2P^2$; $\Delta P = .2P(7.5 - P)$; $P_{t+1} = 2.5P_t - .2P_t^2$

1.2.6. b. The MATLAB commands `x=[0:.1:12]`, `y=x+.8*x.*(1-x/10)`, `plot(x,y)` work.

c. The cobweb diagram should fit well with the table below.

t	0	1	2	3	4	5
P_t	1	1.72	2.8593	4.4927	6.4721	8.2988

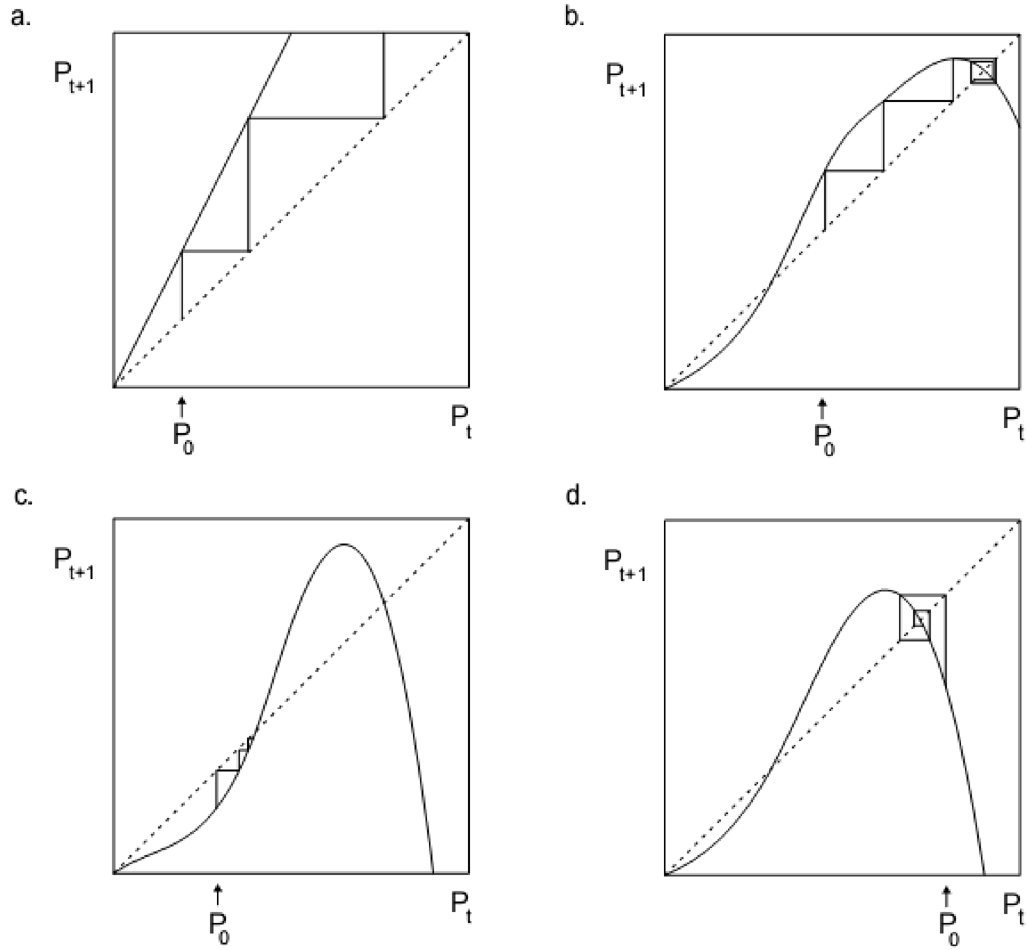
However, it is hard to cobweb very accurately by hand, so you shouldn't be too surprised if your diagram matches the table poorly. Errors tend to compound with each additional step.

1.2.7. After graphing the data, a logistic equation seems like a reasonable choice for the model. Estimating from the table and graph, $K \approx 8.5$ seems like a good choice for the carrying capacity. Since $P_2/P_1 \approx 1.567$, a reasonable choice for r is .567. However, trial and error shows that increasing the r value a bit appears to give an even better logistic fit. Here is one possible answer: $\Delta P = .63P(1 - P/8.5)$.

1.2.8. a. $M_{t+1} = M_t + .2M_t(1 - \frac{M_t}{200})$, where M_t is measured in thousands of individuals. Notice that the carrying capacity is $K = 200$ thousands, rather than $200,000$ individuals. In addition, observe that if the model had been exponential, $M_{t+1} = M_t + .2M_t$, that changing the units would have no effect on the formula expressing the model.

b. $L_{t+1} = L_t + .2L_t(1 - L_t)$, where L_t is measured in units of $200,000$ individuals.

1.2.9.



1.2.11. a. The equation states the change in the amount N of chemical 2 is proportional to the amount of chemical 2 present. Values of r that are reasonable are $0 \leq r \leq 1$ and $N_0 = 0$. (However, if $r = 1$, then all of chemical 1 is converted to chemical 2 in a single time step.) A graph of N_t as a function of t looks like an exponential decay curve that has been reflected about a horizontal axis, and moved upward so that it has a horizontal asymptote at $N = K$. Thus, N_t is an increasing function, but its rate of increase is slowing for all time.

b. The equation states the amount of chemical 2 created at each time step is proportional to both the amount of chemical 1 and the amount of chemical 2 present. This equation describes a discrete logistic model, and the resulting time plot of N_t shows typical logistic growth. Note that with a small time interval, r should be small, and so the model should not display oscillatory behavior as it approaches equilibrium. If N_0 equals zero, then the chemical reaction will not take place, since at least a trace amount of chemical 2 is necessary for this particular reaction. The shape of a logistic curve makes a lot

Allman 1.3, Ex. 6(a)(c),7(a),11

1.3.6. a. $P^* = 0, 15$

b. $P^* = 0, 44$

c. $P^* = 0, 20$

d. $P^* = 0, a/b$

e. $P^* = 0, (c-1)/d$

1.3.7. a. At $P^* = 0$, the linearization is $p_{t+1} \approx 1.3p_t$. Since $|1.3| > 1$, $P^* = 0$ is unstable. At $P^* = 15$, the linearization process gives

$$\begin{aligned}
15 + p_{t+1} &= 1.3(15 + p_t) - .02(15 + p_t)^2 \implies \\
15 + p_{t+1} &= [1.3(15) - .02(15)^2] + 1.3(p_t) - .02(30p_t + p_t{}^2) \implies \\
p_{t+1} &= 1.3(p_t) - .02(30p_t + p_t{}^2) \implies \\
p_{t+1} &\approx 1.3(p_t) - .02(30p_t) = .7p_t.
\end{aligned}$$

Since $|.7| < 1$, $P^* = 15$ is stable.

1.3.11. a. Since the concentration of oxygen in the blood stream can not be more than that of the lung, B can not change by more than half the difference $(L - B)$; thus, $0 < r \leq .5$.

b. $\Delta B = r(K - 2B)$

c. If we choose an initial value $0 < B_0 < .5$ for the oxygen concentration in the bloodstream, then B steadily increases up to $B^* = .5K$. The rate of increase slows as B gets close to $.5K$. If r is increased to values just slightly smaller than .5, then B approaches equilibrium quite quickly, much more quickly than with $r = .1$.

d. $B^* = K/2$. (Note that the denominator is the total volume of the two compartments, and B^* has the correct units.) This answer makes sense in that the equilibrium concentration for B (and for L) would be (amount of oxygen)/(total volume).

e. $\Delta b = r(K - 2(K/2 + b_t)) = -2rb_t$. Equivalently, $b_{t+1} = (1 - 2r)b_t$.

f. $b_t = (1 - 2r)^t b_0$. $B_t = K/2 + (1 - 2r)^t b_0$. Note that $b_0 < 0$, since we assume that $L > B$ initially. So, since $0 \leq 1 - 2r < 1$, B increases up to its equilibrium value of $K/2$.

g. Suppose the volume of the lung is V_L and the volume of the bloodstream is V_B, then the total amount of oxygen $K = LV_L + BV_B$ and $L = (K - BV_B)/V_L$. The equation for ΔB then becomes $\Delta B = r((K - BV_B)/V_L - B)$ and the equilibrium is $B^* = K/(V_L + V_B)$, etc.