

1. Suppose two types of individuals  $A$  and  $B$  occur in the population with relative frequency  $x$  and  $y$ , and with fitness  $f_a$  and  $f_b$  respectively, so that the replicator equations are

$$\begin{aligned}\Delta x &= x(f_a - \phi) \\ \Delta y &= y(f_b - \phi)\end{aligned}$$

where  $\phi$  is a "selection" function, to be determined.

- (a) Prove that in order to maintain  $x+y = 1$  the function  $\phi$  must take the form  $\phi = f_a x + f_b y$ . (Hint: Try adding the two equations above and use the fact that if  $x + y = 1$  then  $\Delta x + \Delta y = 0$ ).

If we add the two equations we get

$$\begin{aligned}\Delta x + \Delta y &= x(f_a - \phi) + y(f_b - \phi) \\ &= x f_a - x \phi + y f_b - y \phi \\ &= f_a x + f_b y - (x + y) \phi\end{aligned}$$

We then make use of the facts that  $x + y = 1$  and  $\Delta x + \Delta y = 0$  to get

$$0 = f_a x + f_b y - \phi \Rightarrow \phi = f_a x + f_b y$$

- (b) Hence show that the above system reduces to the single equation

$$\Delta x = (f_a - f_b)x(1 - x)$$

First we insert  $y = 1 - x$  into the expression for  $\phi$  to get  $\phi = f_a x + f_b(1 - x) = f_b + f_a x - f_b x = f_b + (f_a - f_b)x$

2. Consider a model for growth which assumes that the rate of grow is proportional to the probability that two individuals meet. In this case a suitable model would be

$$\Delta x = ax^2$$

- (a) This model is called the super exponential growth model. Compare the the growth predicted by this model with the exponential growth  $\Delta x = ax$  for the case where  $a = 1$ , for  $t = 0, 1, 2, 3, 4$ , and  $x_0 = 1$ .

$t$	0	1	2	3	4
$p_t$ (exponential)	1	$1 + 1 = 2$	$2 + 2 = 4$	$4 + 4 = 8$	$8 + 8 = 16$
$p_t$ (super exponential)	1	$1 + 1^2 = 2$	$2 + 2^2 = 6$	$6 + 6^2 = 42$	$42 + 42^2 = 1806$

- (b) Now suppose that two types of organisms,  $A$  and  $B$  are in an environment at its carrying capacity, so that the total population is constant. Let  $x$  be the relative number of  $A$

types and  $y$  be the relative number of  $B$  types, so that  $x + y = 1$ . In this case a suitable model for the growth of these two types is:

$$\begin{aligned}\Delta x &= ax^2 - \phi x \\ \Delta y &= by^2 - \phi y\end{aligned}$$

Show that in order to maintain  $x + y = 1$  then  $\phi = ax^2 + by^2$ .

We use the fact that  $x + y = 1$  and  $\Delta x + \Delta y = 0$ . If we add the two equations we get:

$$\Delta x + \Delta y = ax^2 + by^2 - \phi x - \phi y \Rightarrow 0 = ax^2 + by^2 - \phi(x + y) \Rightarrow \phi = ax^2 + by^2$$

- (c) Hence show that the two dimensional system can be rewritten as the one dimensional equation:

$$\Delta x = x(1 - x)(ax - b(1 - x))$$

Take  $y = 1 - x$  in the first equation to get

$$\begin{aligned}\Delta x &= ax^2 - (ax^2 + by^2)x = x(ax - ax^2 - b(1 - x)^2) \\ &= x(ax(1 - x) - b(1 - x)^2) = x(1 - x)(ax - b(1 - x))\end{aligned}$$

- (d) Find all the equilibria for this model and determine their stability (note:  $a > 0$  and  $b > 0$ . It may help for you to choose specific values for  $a$  and  $b$  as an example).

The equilibrium is when  $\Delta x = 0$ . This happens whenever one of the factors is zero, which occurs for  $x = 0$ ,  $x = 1$ , or  $ax - b(1 - x) = 0 \Rightarrow (a + b)x - b = 0 \Rightarrow x = \frac{b}{a+b}$ . To consider stability note that when  $x$  is between 0 and 1 the first two factors are positive, so that the sign of  $\Delta x$  depends only on the third factor  $(a + b)x - b$ . This is a straight line with positive slope  $a + b$  and  $y$ -intercept at  $-b$ . So below the middle equilibrium  $\Delta x < 0$  and above it  $\Delta x > 0$ . Thus the middle equilibrium is unstable and  $x = 0$  and  $x = 1$  are stable.

- (e) Bonus problem for calculus students: show that the solution to the continuous version of the super exponential growth model

$$\dot{x} = ax^2$$

is

$$x = \frac{x_0}{1 - atx_0}$$

and that this solution diverges to infinity in finite time. This is not a good thing! Differentiating the solution using the chain rule gives

$$\dot{x} = - \left( \frac{x_0}{(1 - atx_0)^2} \right) (-ax_0) = \frac{ax_0^2}{(1 - atx_0)^2} = ax^2$$