

1. Suppose two types of individuals A and B occur in the population with relative frequency x and y , and with fitness f_a and f_b respectively, so that the replicator equations are

$$\begin{aligned}\Delta x &= x(f_a - \phi) \\ \Delta y &= y(f_b - \phi)\end{aligned}$$

where ϕ is a "selection" function, to be determined. Prove that in order to maintain $x + y = 1$ the function ϕ must take the form $\phi = f_a x + f_b y$. (Hint: Try adding the two equations above and use the fact that if $x + y = 1$ then $\Delta x + \Delta y = 0$).

2. Consider a model for growth which assumes that the rate of grow is proportional to the probability that two individuals meet. In this case a suitable model would be

$$\Delta x = ax^2$$

- (a) This model is called the super exponential growth model. Compare the the growth predicted by this model with the exponential growth $\Delta x = ax$ for the case where $a = 1$, for $t = 0, 1, 2, 3, 4, 5$, and $x_0 = 1$.
- (b) Now suppose that two types of organisms, A and B are in an environment at its carrying capacity, so that the total population is constant. Let x be the relative number of A types and y be the relative number of B types, so that $x + y = 1$. In this case a suitable model for the growth of these two types is:

$$\begin{aligned}\Delta x &= ax^2 - \phi x \\ \Delta y &= by^2 - \phi y\end{aligned}$$

Show that in order to maintain $x + y = 1$ then $\phi = ax^2 + by^2$.

- (c) Hence show that the two dimensional system can be rewritten as the one dimensional equation:

$$\Delta x = x(1 - x)(ax - b(1 - x))$$

- (d) Find all the equilibria for this model and determine their stability (note: $a > 0$ and $b > 0$. It may help for you to choose specific values for a and b as an example).
- (e) Bonus problem for calculus students: show that the solution to the continuous version of the super exponential growth model

$$\dot{x} = ax^2$$

is

$$x = \frac{x_0}{1 - atx}$$

and that this solution diverges to infinity in finite time. This is not a good thing!