1. Two types of individuals, A and B, have relative frequencies x and y, with $x+y=1$. Type A's have fitness 1 and type B's has fitness f. Suppose B mutates to A with probability 0.1, but A does not mutate to B.

   (a) Write down the two replicator/mutator equations for Δx and Δy.

$$\begin{aligned}\Delta x &= x+\beta y f-(x+fy)x \\ \Delta y &= (1-\beta)yf-(x+fy)y\end{aligned}$$

   (b) Assuming $f=1.2$ find the equilibrium value of type A individuals.
   We use the fact that $y=1-x$ and to get:

$$\Delta x = x+\beta f(1-x)-x^2-f(1-x)x = x(1-x)+\beta f(1-x)-f(1-x)x = (1-x)(x+\beta f-fx)$$

   So there is an equilibrium at $x=1$ and

$$x+\beta f-fx=0 \Rightarrow x^*=\frac{\beta f}{f-1}=\frac{(0.1)(1.2)}{0.2}=0.6$$

   (c) Find an expression for the equilibrium number of type A individuals as a function of f and state the conditions on f so that this equilibrium is between 0 and 1.
   From the above expression

$$x^*=\frac{\beta f}{f-1}$$

   $x^*>0$ means $f>1$ and $x^*<1$ means

$$f\beta<f-1 \Rightarrow f(1-\beta)>1 \Rightarrow f>\frac{1}{1-\beta}=\frac{1}{0.9}=1.\dot{1}$$

   (d) For which fitness value does the equilibrium result in an even distribution of the two types?
   We set $x^*=\frac{1}{2}$ so that

$$2\beta f=f-1 \Rightarrow f=\frac{1}{1-2\beta}=\frac{1}{0.8}=1.25$$

2. Three types of individuals, A, B and C, have relative frequencies x, y and z. Suppose they each have equal fitness equal to 1. Suppose A's mutate to B's with probability 0.1 but do not mutate to C's; B's mutate to C's with probability 0.2, but do not mutate to A's, and C's mutate to A's with probability 0.3, but do not mutate to B's

(a) Write down the mutation matrix for this system.

$$Q = \begin{pmatrix} 0.9 & 0 & 0.3 \\ 0.1 & 0.8 & 0 \\ 0 & 0.2 & 0.7 \end{pmatrix}$$

(b) Write down the dynamical equations for Δx, Δy and Δz.

$$\begin{aligned} \Delta x &= -0.1x + 0.3z \\ \Delta y &= 0.1x - 0.2y \\ \Delta z &= 0.2y - 0.3z \end{aligned}$$

(c) Find the equilibrium distribution of the different types of individuals.
We use $z = 1 - x - y$ in the first two equations and set $\Delta x = 0$ and $\Delta y = 0$.

$$\Delta x = 0 \;\Rightarrow -0.1x + 0.3(1 - x - y) = 0 \;\Rightarrow 0.4x + 0.3y + 0.3$$

and

$$\Delta y = 0 \;\Rightarrow 0.1x - 0.2y = 0 \;\Rightarrow\; x = 2y$$

In the first equation this leads to

$$0.8y + 0.3y = 0.3 \;\Rightarrow\; y = \frac{3}{11}\,, \quad x = \frac{6}{11}\,, \quad z = \frac{2}{11}$$

(d) Use Excel to find and plot the population for the first 20 time steps, assuming the initial population consists entirely of type A individuals.

3. In game theory, A and B are called strategies because they often correspond to types of behavior that individuals might choose to take when they interact (eg fight or not fight, share or not share etc). An evolutionarily stable strategy (ESS) is one which will oppose the invasion of a rare mutant strategy. Suppose in a large population of type A individuals a rare mutant of type B is introduced. Then A will be an ESS if one of the two situations arises: (i) $a > c$ or (ii) $a = c$ and $b > d$. When B is rare it will most likely meet another A. If condition (i) holds B will be less fit then A and die out. If condition (ii) holds B will have the same fitness as an A when they interact with other A's. However, in this case the rare circumstances when they both meet another B will be important. Then the second part of condition (ii) will guarantee that B is less fit than A from those interactions. For each of the following matrices, decide whether one, both or none of the strategies are ESS, and whether any of the games allow both strategies to coexist.

(a)

	A	B
A	2	1
B	3	5

(b)

	A	B
A	5	3
B	5	2

(c)

	A	B
A	5	1
B	2	3

(d)

	A	B
A	2	5
B	3	1

For (a) A is not ESS because $2 < 3$, but B is ESS, because $5 > 1$. For (b) A is ESS, because although the first column has identical entries, the second column has $3 > 2$. B is not ESS. For (c) A is ESS because $5 > 2$, and B is also ESS because $3 > 1$. This is a bistable situation. For (d) neither is ESS, but there is a stable coexistence.

4. For which range of values of the parameter p does the following game have a stable coexistence of A and B? Find the equilibrium value of the population of A individuals as a function of p. For which value of p will the population be equally distributed between A and B individuals?

	A	B
A	2	5
B	p	0

For equilibrium to be possible we need B to be fittest when A is common and A to be fittest when B is common. Since $5 > 0$, the second condition is satisfied, Thus we need $p > 2$ to satisfy the first condition. Equilibrium is when the fitness of A and B are equal. Hence $f_A = f_B \Rightarrow 2x + 5y = px + 0$. If we put $y = 1 - x$ we get:
$2x + 5(1-x) = px \Rightarrow -3x + 5 = px \Rightarrow 5 = (p+3)x \Rightarrow x = 5/(p+3)$. Equal distribution is when $x = \frac{1}{2}$, which means $p + 3 = 10$. Thus $p = 7$.

5. In the Hawk-Dove game two individuals compete for a resource with a fitness benefit b (perhaps food, a home, or a mate). An individual using the Hawk strategy postures first but escalates to a fight if threatened. An individual using the Dove strategy postures first, but retreats if threatened. Individuals who fight and lose incur a cost c and get no benefit. Those who fight and win incur no cost and get benefit b. When a individual using the Hawk strategy meets another Hawk imagine that half the time they lose and half the time they win. Similarly, when two Dove players meet imagine that half the time they get the benefit and half the time they don't. Then a good matrix describing this game is:

	H	D
H	$\frac{1}{2}(b-c)$	b
D	0	$\frac{1}{2}b$

(a) Assuming b and c are both bigger than zero can Dove ever be an ESS?
No. For Dove to be an ESS we need $\frac{1}{2}b \geq b$. This can only happen if $b \leq 0$ which does not make sense for this game. A negative benefit is not a benefit!

(b) What is the condition on the cost c for Hawk to be an ESS?
Here we need $\frac{1}{2}(b-c) \geq 0$ which means $c \leq b$. So if the cost of fighting is less than the benefit of the resource it always pays to fight!

(c) What happens to the population of Hawks and Doves if c does not satisfy this condition?
If $c > b$ then Doves can invade a Hawk only strategy. However, Doves do not take over since they are not an ESS. Instead there is a stable equilibrium of Hawks and Doves.

(d) What are the values of b and c corresponding to the Hawk-Dove reward matrix below?

	H	D
H	-2	2
D	0	1

By direct comparison $b = 2$ and $\frac{1}{2}(b - c) = -2$ which means $2 - c = -4 \Rightarrow c = 6$.

(e) Show Hawk and Dove can coexist and find the expected equilibrium distribution of Hawks and Doves.

Since $c > b$ we have the condition stated in part (c) for an equilibrium. This occurs when $f_H = f_D \Rightarrow -2x + 2y = 0x + y \Rightarrow y = 2x \Rightarrow 1 - x = 2x \Rightarrow 1 = 3x \Rightarrow x = \frac{1}{3}$. So a population where one third play a Hawk-like strategy and two thirds play a Dove-like strategy is stable.