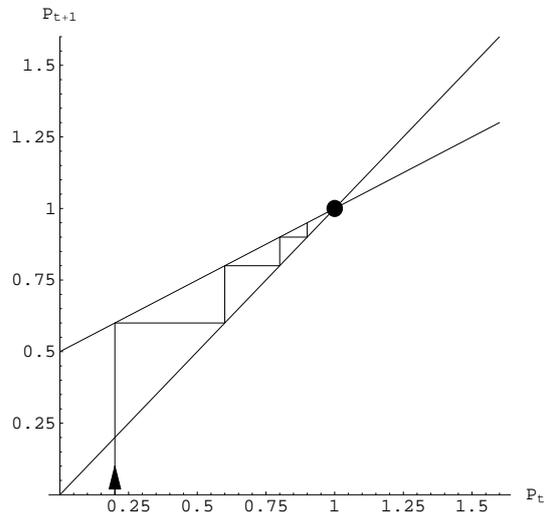
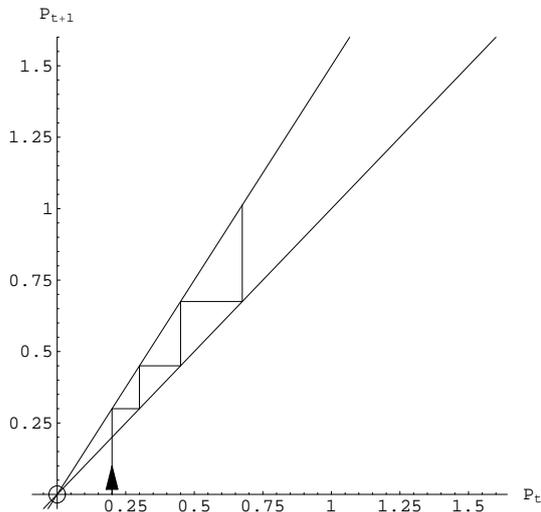
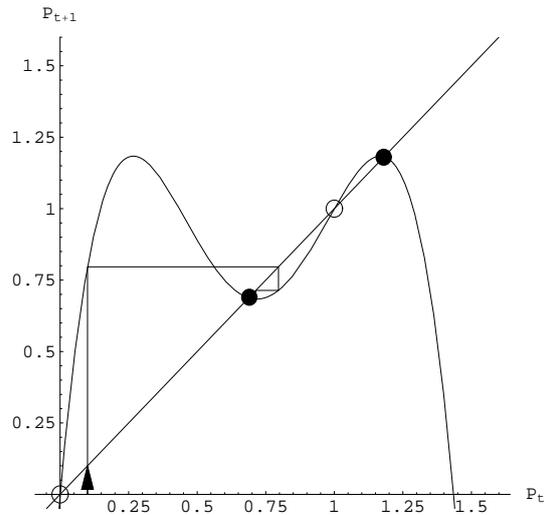
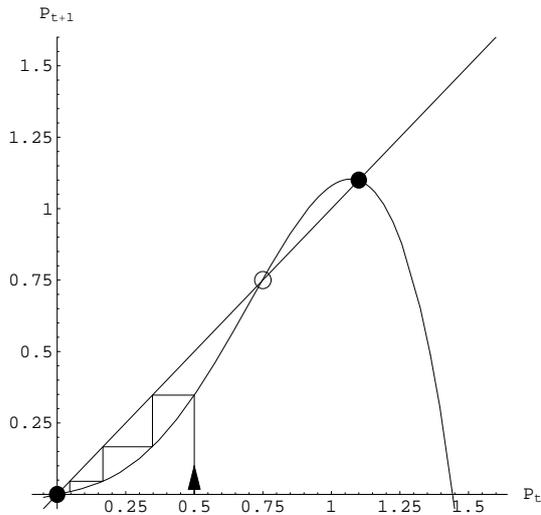


1. For the difference equation  $M_{t+1} = 0.5M_t + 1$ ,
  - (a) find  $M_t$  at  $t=1,2,3,4$  and 5 if the initial value is  $M_0 = 10$ .  
 $M_1 = 0.5M_0 + 1 = 0.5(10) + 1 = 6$ .  $M_2 = 0.5M_1 + 1 = 0.5(6) + 1 = 4$ . Similarly  $M_3 = 3$ ,  $M_4 = 2.5$ , and  $M_5 = 2.25$ .
  - (b) What is/are the equilibrium value/s of this model? Based on your calculations in (a), is the equilibrium stable or unstable?  
 $M_t$  reaches and equilibrium of 2, which is stable.
  - (c) Find an expression for  $M_t$  as a function of time  $t$ . (Hint: Consider,  $M_t = M^* + u_t$  and find an exact solution for  $u_t$ , given the initial starting point.)  
 This is exponential decay down to the value 2.  
 $M_0 = 8 + 2$ ,  $M_1 = 4 + 2$ ,  $M_2 = 2 + 2$ ,  $M_3 = 1 + 2$ ,  $M_4 = 0.5 + 2$ , and  $M_5 = 0.25 + 2$ .  
 The pattern is  $M_t = 8(0.5)^t + 2$ .
  
2. A female mouse reproduces approximately once every two months for the duration of her life which is approximately one year. She produces a litter size of about 8 (4 males and 4 females) on average, under ideal conditions.
  - (a) Estimate the birth rate, death rate and per capita growth rate of mice choosing months as your time unit. Discuss your assumptions and answers with your group.  
 In her life time a female produces about 4 females every two months for a year, which is 24 females a year, so the birth rate is  $24/12=2$  per month. The death rate is approximately the reciprocal of the life span which is  $(1/12)=0.0833$  per month. Thus the per capita growth rate is  $r = b - d = 2 - 0.833 \approx 1.9$ .
  - (b) Assuming the per capita growth rate from above estimate how long it would take a group of 10 to grow to a population size of 1000, assuming no limits on growth due to competition for resources.  
 The difference equation is  $P_{t+1} = (1 + r)P_t = 2.9P_t$ , which means that the general solution is  $P_t = 10(2.9)^t$ .  
 The population reaches 1000 when  $1000 = 10(2.9)^t \Rightarrow 100 = (2.9)^t \Rightarrow t = \ln(100)/\ln(2.9) \approx 4.3$  months.
  - (c) Now assume that there are limits on growth and that the carrying capacity for a particular environment is 2000 mice. Write down the difference equation for the logistic model that would describe how the population changes.  
 The carrying capacity is 2000 and the intrinsic growth rate is 1.9, so the Logistic model is  $P_{t+1} = P_t + 1.9P_t(1 - P_t/2000)$ .

3. For each of the following graphs showing  $x_{t+1}$  vs  $x_t$  for some population label all the equilibrium points, indicate their stability, and draw a cobweb diagram showing the population for four time steps starting at the point indicated



The equilibria occur where the function crosses the line  $y = x$ . These are indicated with dots on the graphs, with a solid dot being stable (cobwebs would approach it) and a circle being unstable (cobwebs would go away from it). When drawing cobwebs be sure to go up to the function first to get your output and then across to the line  $y = x$  to make your output the next input.

4. In the absence of predation, a population of fish grows according to the logistic difference equation

$$\Delta P = 0.2P(1 - P)$$

where  $P$  is measured in units so that the carrying capacity is 1. Suppose fish are harvested at a rate proportional to the number of fish so that the new difference equation is

$$\Delta P = 0.2P(1 - P) - hP$$

where  $h$  is a proportionality constant indicating the rate of harvesting.

- (a) What is the biologically meaningful range of values for  $h$ ?

$h$  is the per capita rate at which the population is removed, so  $h$  can be no larger than 1 (you can't remove more than 100%) and  $h$  must be positive, otherwise you are not removing fish.

- (b) Find the equilibrium points for the population with harvesting, expressing your answer in terms of  $h$ .

Set  $\Delta P = 0$  and solve for  $P$ . This gives

$$0.2P(1-P) - hP = 0 \Rightarrow P(0.2 - h - 0.2P) = 0 \Rightarrow P = 0 \quad \text{or} \quad P = \frac{0.2 - h}{0.2} = 1 - \frac{h}{0.2} = 1 - 5h.$$

The second equilibrium value is reduced from the carrying capacity of 1 for positive values of  $h$ .

- (c) Determine the harvesting level for which the only equilibrium is at  $P = 0$ . This will happen when the second fixed point is  $P = 0$ , which occurs when  $5h = 1 \Rightarrow h = 0.2$ . This makes sense since the intrinsic growth rate is 0.2.

- (d) Determine the stability of the equilibria (Hint: this will depend on the value of  $h$  is larger or smaller than the value calculated above.).

For stability we look at the second fixed point  $P = 1 - \frac{h}{0.2}$ . When this is bigger than the fixed point  $P = 0$ , then  $P = 0$  is unstable and  $P = 1 - \frac{h}{0.2}$  is stable. This happens when  $h > 0.2$ . For  $h < 0.2$  the second fixed point is negative (and unphysical). The fixed point at  $P = 0$  becomes stable – all solutions tend to  $P = 0$ .